Exercise #2

January 24, 2023

Problem 1.

a) Show that a (not necessarily differentiable) function $f\colon \mathbb{R}^n \mapsto \mathbb{R}_{>0}$ is convex, if the function $x \mapsto \log(f(x))$ is convex.

b) Show that an optimization problem $\min_{x\in\mathbb{R}^n} f(x)$ has at most one global minimizer if the objective function $f\colon \mathbb{R}^n \mapsto \mathbb{R}$ is strictly convex. In addition, find a strictly convex objective function f that has no global minimizer at all.

Problem 2.

Show that the function $f\colon \mathbb{R}^2 \mapsto \mathbb{R}$,

$$f(x,y) = \log\bigl(e^x + e^y\bigr)$$

is convex.

Problem 3.

Consider the optimization problem

$$\min_{x\in\mathbb{R}^n} f(x),$$

where the objective function $f\colon \mathbb{R}^3 \mapsto \mathbb{R}$ is defined as

$$f(x,y,z) = 2x^2 + xy + y^2 + yz + z^2 - 6x - 7y - 8z + 9.$$

Prove that this optimization problem has a unique global minimizer and find it.

Problem 4.

Consider the function $f\colon \mathbb{R}^2 \mapsto \mathbb{R}$ (see Exercise 1, Problem 3a)

$$f(x,y) = \frac{x^2}{2} + x\cos y.$$

We want to perform one step of a line search method with initial value $x_0 = (1, \frac{\pi}{4})$ and search direction $p_0 = (-1, 0)$.

a) Confirm that p_0 is a descent direction from the initial point x_0.

b) State the Armijo condition. What is the range of admissible values for the step length α, if a parameter $c = 0.1$ is used?

c) Perform one step of the line search method using the optimal value of α as step length.

Problem 5.

a) Consider the function $f\colon \mathbb{R}^2 \mapsto \mathbb{R}$ (see Exercise 1, Problem 3b),

$$f(x, y) = 2x^2 - 4xy + y^4 + 5y^2 - 10y.$$

Perform one step of the gradient descent method with backtracking (Armijo) line search starting from the point $x_0 = (0, 0)$. Start with an initial step lenght $\alpha = 1$ and use the parameters $c = 0.1$ (sufficient decrease parameter) and $\rho = 0.1$ (contraction factor).

b) Consider the function $f\colon \mathbb{R}^2 \mapsto \mathbb{R}$,

$$f(x, y) = x^4y^2 + x^4 - 2x^3y - 2x^2y - x^2 + 2x + 2.$$

Perform one step of the gradient descent method with backtracking (Armijo) line search starting from the point $x_0 = (0, 0)$. Start with an initial step length $\alpha = \frac{1}{2}$ and use the parameters $c = \frac{1}{2}$ (sufficient decrease parameter) and $\rho = 0.1$ (contraction factor).

Problem 6.

a) Assume that the sequence $\{x_k\}_{k\in\mathbb{N}}$ is generated by the gradient descent method with backtracking (Armijo) line search for the minimization of a function f, and that $\nabla f(x_k) \neq 0$ for all k. Moreover, assume that $\overline{x}$ is an accumulation point of the sequence $\{x_k\}_{k\in\mathbb{N}}$. Show that $\overline{x}$ is not a local maximum of f.

b) We consider a line search method of the form $x_{k+1} = x_k + \alpha_k p_k$ for the minimization of the function $f\colon \mathbb{R}^n \mapsto \mathbb{R}$ with the search direction p_k given as

$$p_k = -\operatorname{sgn}\big((\nabla f(x_k))_i\big)e_i,$$

where the index i is chosen such that $|(\nabla f(x_k))_i|$ is maximal. Here e_i with $1 \le i \le n$ denotes i^{th} standard basis vector in $\mathbb{R}^n$. Show that the direction p_k is a descent direction whenever x_k is not a stationary point of f (that is, $\nabla f(x_k) \neq 0$).