Exercise #12

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Problem 1.

Find (and simplify, if possible) the dual of the linear programme

$$\min c^T x \qquad \text{subject to } Ax \geq b,\ x \geq 0.$$

Problem 2.

Find the dual of the linear optimisation problem

$$5x_1 + 3x_2 + 4x_3 \longrightarrow \min \qquad \text{subject to } \begin{cases} x_1 + x_2 + x_3 = 1, \\ \qquad\qquad x_i \geq 0, \quad i = 1, 2, 3, \end{cases}$$

and compute its (i.e., the *dual's*) solution. Use the dual's solution in order to find a solution of the original problem.

Problem 3.

Assume that $A \in \mathbb{R}^{m\times n}$ with $m < n$ is a matrix of full rank and that $b \in \mathbb{R}^m \setminus \{0\}$. Consider the optimization problem

$$\frac{1}{2}\|x\|^2 \longrightarrow \min \qquad \text{subject to} \qquad Ax = b. \tag{1}$$

a) Compute the solution of (1).

b) Derive an explicit formula for the dual problem to (1).

c) Show that $\lambda^* \in \mathbb{R}^m$ solves the dual problem, if and only if

$$AA^T\lambda^* = b.$$

d) Verify directly that in this situation

$$\min_{x\in\mathbb{R}^n} \max_{\lambda\in\mathbb{R}^m} \mathcal{L}(x,\lambda) = \max_{\lambda\in\mathbb{R}^m} \min_{x\in\mathbb{R}^n} \mathcal{L}(x,\lambda).$$

Problem 4.

Consider the linear programme

$$c^T x \longrightarrow \min \qquad \text{subject to } Ax = b \text{ and } x \geq 0,$$

where $c \in \mathbb{R}^d$, $A \in \mathbb{R}^{m\times d}$, and $b \in \mathbb{R}^m$. We can approximate this problem using a logarithmic barrier function, which results in the new problem

$$f(x) - \beta \sum_i \ln(x_i) \qquad \text{subject to } Ax = b. \tag{2}$$

Here we set $-\ln(t) = +\infty$ for $t \leq 0$.

Compute the Lagrangian dual of (2).