Exercise #11

April 18, 2023

Problem 1.

Decide for each of the following binary relations whether it is a partial order, a total order, or no order relation at all.

a) The relation $\preceq$ on $\mathbb{C}$ given by $x \preceq y$ if $\Re x \le \Re y$ (here $\Re x$, $\Re y$ denote the real part of x and y, respectively).

b) The relation $\preceq$ on $\mathbb{R}^n$, $n \ge 1$, given by $x \preceq y$ if $\|x\| \le \|y\|$.

c) The relation $\preceq$ on $\mathbb{R}^n$, $n \ge 1$, given by $x \preceq y$ if $x_1 \le y_1$ and $x_i = y_i$ for $2 \le i \le n$.

d) The relation $\preceq$ on the set of cubic polynomials given by $p \preceq q$ if the largest roots x_p, x_q of p and q, respectively, satisfy $x_p \le x_q$.

Problem 2.

On the space $\mathbb{R}^d$ we can define the relation $x \le_{\text{lex}} y$ if either $x = y$ or there exists $1 \le i \le d$ such that $x_j = y_j$ for $j < i$ and $x_i < y_i$.

a) Show that $\le_{\text{lex}}$ defines a total order on $\mathbb{R}^d$ (the *lexicographical order*).

b) Show that the space $(\mathbb{R}^d, \le_{\text{lex}})$ is an ordered vector space.

c) Sketch the cone $C := \{x : 0 \le_{\text{lex}} x\}$ in the case $d = 2$.

Problem 3.

Define the functions $f_1 \colon \mathbb{R} \to \mathbb{R}$, $f_1(x) = x^2$ and $f_2 \colon \mathbb{R} \to \mathbb{R}$, $f_2(x) = (x^2 - 1)^2$, and consider the multicriteria optimisation problem

$$\min_{x \in \mathbb{R}} \big(f_1(x), f_2(x)\big). \tag{1}$$

a) Sketch the image $Y := \{(f_1(x), f_2(x)) : x \in \mathbb{R}\} \subset \mathbb{R}^2$ of (f_1, f_2) and find all minimal points in Y.

b) Find all Pareto-optimal solutions of (1).

Problem 4.

Define the functions $f_1, f_2\colon \mathbb{R}^2 \to \mathbb{R}$

$$f_1(x, y) = \frac{1}{x^4 + y^4 + 1}, \qquad f_2(x, y) = x^2 + y^2,$$

and consider the multicriteria optimisation problem

$$\min_{(x,y)\in\mathbb{R}^2} \big(f_1(x, y), f_2(x, y)\big). \tag{2}$$

a) Sketch the image $Y := \{(f_1(x, y), f_2(x, y)) : (x, y) \in \mathbb{R}^2\} \subset \mathbb{R}^2$ of (f_1, f_2) and find all minimal points in Y.

b) Show that the Pareto-optimal solutions of (2) are precisely the points of the form $(x^*, 0)$, $x^* \in \mathbb{R}$, and $(0, y^*)$, $y^* \in \mathbb{R}$.

c) Show that there does not exist any $0 \le \lambda \le 1$ such that $(x^*, 0) = (1/2, 0)$ is a solution of the weighted sum problem

$$\min_{(x,y)\in\mathbb{R}^2} \Big(\lambda f_1(x, y) + (1 - \lambda) f_2(x, y)\Big).$$