Exercise #10

March 21, 2023

Problem 1. (*Exercise 16.1 in N&W*)

Consider the quadratic programme

$$f(x, y) := 2x + 3y + 4x^2 + 2xy + y^2 \longrightarrow \min$$

subject to

$$x - y \geq 0, \qquad x + y \leq 4, \qquad x \leq 3.$$

a) Solve the quadratic program and sketch its geometry (that is, the domain of the problem and the level lines of the function f).

b) What happens if one replaces the function f by $-f$? Does the problem still have solutions or local solutions?

Solution.

a) We can see that the problem is a quadratic minimization problem

$$\min \frac{1}{2} X^\top G X + c^\top X \quad \text{s.t.} \quad a_i^\top X - b_i \geq 0,$$

where

$$G = \begin{bmatrix} 8 & 2 \\ 2 & 2 \end{bmatrix}, \quad c = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad a_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad a_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

and $X = (x, y)^T$, $b_1 = 0$, $b_2 = -4$ and $b_3 = -3$. We can check that G is positive definite, so by Theorem 16.4 in N&W, the KKT conditions are necessary and sufficient for minimizers. We therefore set up the KKT conditions:

$$8x + 2y + 2 - \lambda_1 + \lambda_2 + \lambda_3 = 0 \tag{1a}$$
$$2x + 2y + 3 + \lambda_1 + \lambda_2 = 0 \tag{1b}$$
$$\lambda_1(x - y) = 0 \tag{1c}$$
$$\lambda_2(4 - x - y) = 0 \tag{1d}$$
$$\lambda_3(3 - x) = 0 \tag{1e}$$
$$x - y \geq 0 \tag{1f}$$
$$4 - x - y \geq 0 \tag{1g}$$
$$3 - x \geq 0. \tag{1h}$$

We see from (1a) and (1b) that

$$x = \frac{1}{6} + \frac{1}{3}\lambda_1 - \frac{1}{6}\lambda_3$$
$$y = -\frac{5}{3} - \frac{5}{6}\lambda_1 - \frac{1}{2}\lambda_2 + \frac{1}{6}\lambda_3.$$

Now, we can go through the usual procedure of considering all options for active constraints. With no active constraints, i.e. $\lambda_1 = \lambda_2 = \lambda_3 = 0$, we get $(x, y) = (\frac{1}{6}, -\frac{5}{3})$ which is, in fact, a KKT point and as such a global solution of the problem. We should end the search for a minimum here, since the problem is strictly convex and the minimizer is unique, as confirmed by Figure 1. However, since the following discussion will prove useful in part b), we carry on looking for KKT points.

Next, we consider cases where only one constraint is active.

First, if $\lambda_1 = \lambda_2 = 0$, i.e. $3 - x = 0$, we get $\lambda_3 = -17$ and $(x, y) = (3, -\frac{9}{2})$. Since $\lambda_3 < 0$, this is not minimizer but a candidate for a maximizer.

Next, if $\lambda_1 = \lambda_3 = 0$, i.e. $4 - x - y = 0$, we get $\lambda_2 = -11$ and $(x, y) = (\frac{1}{6}, \frac{23}{6})$, which breaks constraint (1f).

Lastly, if $\lambda_2 = \lambda_3 = 0$, i.e. $x - y = 0$, we get $\lambda_1 = -\frac{11}{7}$ and $(x, y) = (-\frac{5}{14}, -\frac{5}{14})$. It is a feasible point, but has a negative Lagrange multiplier, meaning it is a candidate for a maximizer. This will prove useful in part b).

Next, we consider cases where two constraints are active.

First, if $\lambda_1 = 0$, i.e. $3 - x = 0$ and $4 - x - y = 0$, we get $(x, y) = (3, 1)$ with corresponding Lagrange multipliers $\lambda_2 = -11$ and $\lambda_3 = -17$, meaning it is not a KKT point but a candidate of maximizer.

Next, if $\lambda_2 = 0$, i.e. $3 - x = 0$ and $x - y = 0$, we get $(x, y) =$ and $(x, y) = (3, 3)$, which breaks constraint (1g).

Lastly, if $\lambda_3 = 0$, i.e. $x - y = 0$ and $4 - x - y = 0$, we get $(x, y) = (2, 2)$ with corresponding Lagrange multipliers $\lambda_1 = \frac{11}{2}$ and $\lambda_2 = -\frac{33}{2}$, meaning it is not a KKT point.

There are no points in which all three constraints are active. Thus, we have one candidate for a minimizer, $(x, y) = (\frac{1}{6}, -\frac{5}{3})$, which is the global minimizer. Figure 1 shows the feasible domain and the contour lines of the objective function which confirm our observations.

b) Replacing f by $-f$ will turn minima into maxima and vice versa. Especially of note is that since $f \to \infty$ as $x^2 + y^2 \to \infty$, then $-f \to -\infty$, meaning there is no global solution to the minimization problem. However, we found three candidates for maximizer $(3, -\frac{9}{2})$, $(-\frac{5}{14}, -\frac{5}{14})$ and $(3, 1)$ in the last problem. The two candidates $(3, -\frac{9}{2})$ and $(-\frac{5}{14}, -\frac{5}{14})$ cannot be maximizer because they are not on the vertex, see figure 1. Only $(3, 1)$ lies on the vertex and hence it is a local maximizer.

Problem 2. (*Exercise 13.1 in N&W*)

Convert the following linear program to standard form:

$$\max_{x,y}(c^T x + d^T y) \text{ subject to } A_1 x = b_1,\ A_2 x + B_2 y \leq b_2,\ l \leq y \leq u, \tag{2}$$

where there are no explicit bounds on x.

Solution.

We want to rewrite (2) in the following standard form

$$\min_z e^T z \text{ subject to } Az = b,\ z \geq 0. \tag{3}$$

First, we turn (2) into a minimization problem,

$$\max_{x,y}(c^T x + d^T y) = \min_{x,y}(-c^T x - d^T y).$$

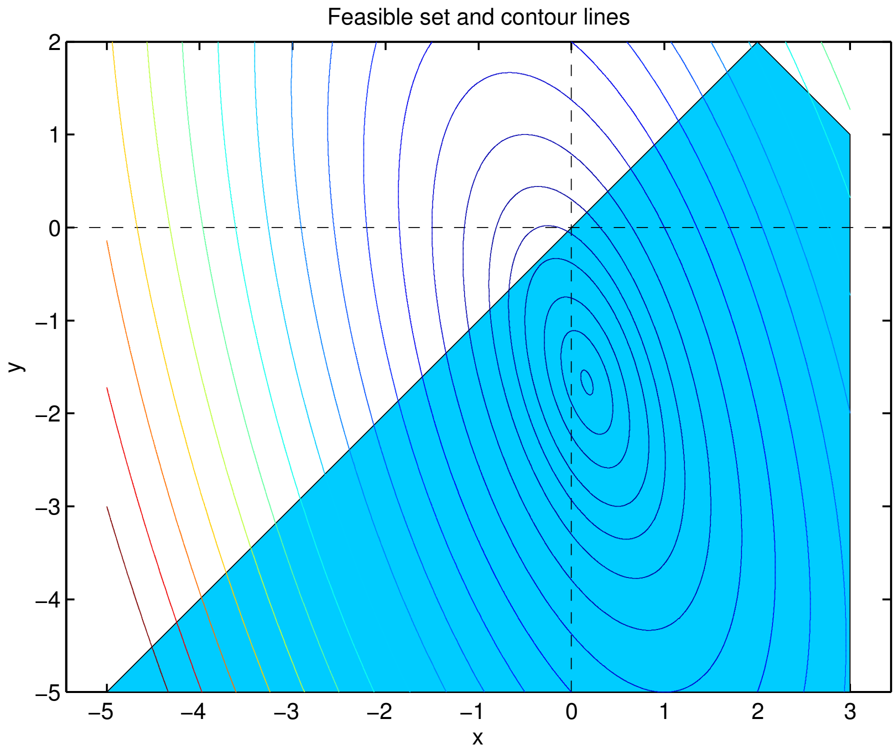

Figure 1: Feasible set (light blue) and contour lines of the function. Note: The feasible set extends further toward infinity.

The first constraint is already an equality constraint, so we keep it for now. For the other constraints, we define slack variables $r, t \geq 0$ and a surplus variable $s \geq 0$, so that the constraints can be written as

$$\begin{aligned} A_1 x &= b_1, \\ A_2 x + B_2 y + r &= b_2, \\ y - s &= l, \\ y + t &= u. \end{aligned}$$

The next trick is to split x and y into nonnegative and nonpositive parts,

$$x = x^+ - x^- \text{ where } x^+ = \max(x, 0) \geq 0,\ x^- = \max(-x, 0) \geq 0.$$

$$y = y^+ - y^- \text{ where } y^+ = \max(y, 0) \geq 0,\ y^- = \max(-y, 0) \geq 0.$$

We can now write our original system (2) in standard form (3) with

$$z = \begin{pmatrix} x^+ \\ x^- \\ y^+ \\ y^- \\ r \\ s \\ t \end{pmatrix},\ e = \begin{pmatrix} -c \\ c \\ -d \\ d \\ 0 \\ 0 \\ 0 \end{pmatrix},\ A = \begin{pmatrix} A_1 & -A_1 & 0 & 0 & 0 & 0 & 0 \\ A_2 & -A_2 & B_2 & -B_2 & I & 0 & 0 \\ 0 & 0 & I & -I & 0 & 0 & I \\ 0 & 0 & I & -I & 0 & -I & 0 \end{pmatrix},\ b = \begin{pmatrix} b_1 \\ b_2 \\ u \\ l \end{pmatrix}.$$

Problem 3.

Assume that $f\colon \mathbb{R}^d \to \mathbb{R}$ is strictly concave and consider the optimization problem

$$\min_x f(x) \text{ subject to } Ax \geq b, \tag{4}$$

where $A \in \mathbb{R}^{m\times d}$ and $b \in \mathbb{R}^m$. Assume that x^* is a local solution of (4). Show that x^* is a vertex of the polyhedron $P = \{x \in \mathbb{R}^d : Ax \geq b\}$.

Solution.

We assume to the contrary that x^* is not the vertex of the polyhedron $P = \{x \in \mathbb{R}^d : Ax \geq b\}$. It follows that there exists $u \neq v \in P$ and $\lambda \in (0,1)$ such that

$$x^* = \lambda u + (1-\lambda)v.$$

Note that P is convex and thus the whole line segment $\{\lambda u + (1-\lambda)v : \lambda \in (0,1)\}$ is contained in P. Thus, by replacing u and v with suitable other points on this line segment, we may assume without loss of generality that $x^* = \frac{1}{2}(u+v)$. Now let $p = \frac{1}{2}(v-u)$ and define $v_\epsilon = x^* + \epsilon p$, $u_\epsilon = x^* - \epsilon p$. Then $v_\epsilon \mapsto x^*$ and $u_\epsilon \mapsto x^*$ as $\epsilon \mapsto 0$ and $x^* = \frac{1}{2}(u_\epsilon + v_\epsilon)$ for all ϵ. The strict concavity of the function f implies that

$$f(x^*) > \frac{1}{2}(f(u_\epsilon) + f(v_\epsilon)),\ \forall\, \epsilon > 0.$$

In particular, for each $\epsilon > 0$ at least one of the inequalities $f(x^*) > f(u_\epsilon)$ or $f(x^*) > f(v_\epsilon)$ holds. Choose therefore for $\epsilon > 0$ some $z_\epsilon \in \{u_\epsilon, v_\epsilon\}$ such that $f(x^*) > f(z_\epsilon)$. By construction $z_\epsilon \mapsto x^*$ and $f(x^*) > f(z_\epsilon)$ for all $\epsilon > 0$.
Moreover, z_ϵ lies on the line segment $\{\lambda u + (1-\lambda)v : \lambda \in [0,1]\}$ for $0 < \epsilon \leq 1$, and thus $z_\epsilon \in P$ for $0 < \epsilon \leq 1$. Together this is a contradiction to the assumption that x^* is a local solution of $\min_{x\in P} f(x)$. Therefore, x^* has to be a vertex of P. $\square$