Chapter 4

Vector Programming

Book: Johannes Jahn: Vector Optimization. Theory, Applications, and Extensions. Springer, 2011.

Example 4.0.0.1. *Assume that you want to buy a bicycle. The two objectives are:*

- *minimal price;*
- *minimal weight.*

Clearly, these objectives are contradicting, as there is most likely no bicycle that has the smallest price as well as the smallest weight (otherwise, the problem would be easy to solve). The question is: How can we compare the bicycles in terms of these both objectives? The issue is the non-totality of the order in $\mathbb{R}^n$*.*

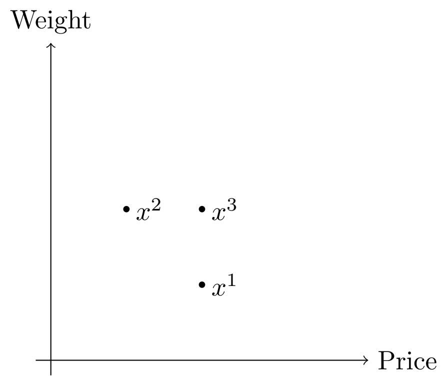

Figure 4.1: x^1 and x^2 are - intuitively - good options for bicycles. The bicycle x^3 is pricier than x^2, with the same weight, and heavier than x^1, with the same price. Therefore, x^3 would not be a good option.

4.1 Optimality Notions

For modeling the notion of optimality in a real linear space (vector space, for example $\mathbb{R}^n$), we need to recall the notion of cones:

Definition 4.1.1 (Cone)**.** *Let C be a nonempty subset of a real linear space X.*

1. *The set C is called a **cone** if*
$$x \in C, \lambda \geq 0 \quad \Longrightarrow \quad \lambda x \in C.$$
2. *A cone C is called **pointed** if*
$$C \cap (-C) = \{0_X\}.$$

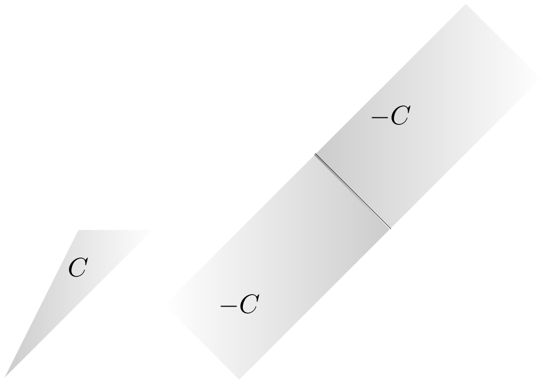

Figure 4.2: Left: C is pointed. Right: C is not pointed.

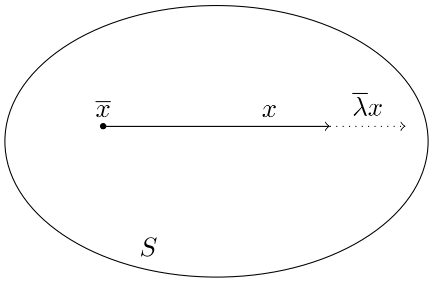

Figure 4.3: $\overline{x} \in \text{core}(S)$.

Definition 4.1.2 (Core)**.** *The set*

$$\text{core}(S) := \{\overline{x} \in S | \forall x \in X \exists \overline{\lambda} > 0 \ s.t. \ \forall \lambda \in [0, \overline{\lambda}] : \ \overline{x} + \lambda x \in S\}$$

is called the ***algebraic interior*** *of S (or the* ***core*** *of S).*

In the following we will often talk about **partial orders**; hence, we first define this term.

Definition 4.1.3. *Let X be a real linear space.*

1. *Each nonempty subset R of the product space $X \times X$ is called a* ***binary relation*** *R on X (we write xRy for $(x, y) \in R$).*
2. *Every binary relation $\leq$ on X is called a partial ordering on X if the following axioms are satisfied (for arbitrary $w, x, y, z \in X$):*
   (a) *$x \leq x$ (reflexive);*
   (b) *$x \leq y,\ y \leq z \Longrightarrow x \leq z$ (transitive);*
   (c) *$x \leq y,\ w \leq z \Longrightarrow x + w \leq y + z$ (compatibility with addition);*
   (d) *$x \leq y,\ \alpha \in \mathbb{R}_+ \Longrightarrow \alpha x \leq \alpha y$ (compatibility with nonnegative scalar multiplication).*
3. *A partial ordering $\leq$ on X is called* ***antisymmetric*** *if for all $x, y \in X$:*

$$x \leq y \text{ and } y \leq x \quad \Longrightarrow \quad x = y.$$

Definition 4.1.4 (Partially Ordered Linear Space)**.** *A real linear space equipped with a partial ordering is called a* ***partially ordered linear space****.*

It is important to note that in a partially ordered linear space two arbitrary elements cannot be compared, in general, in terms of the partial ordering. A significant characterization of a partial ordering in a real linear space is given by

Theorem 4.1.5. *Let X be a real linear space.*

1. *If $\leq$ is a partial ordering on X, then the set*

$$C := \{x \in X | 0_X \leq x\}$$

is a convex cone. If, in addition, $\leq$ is antisymmetric, then C is pointed.

2. *If C is a convex cone in X, then the binary relation*

$$\leq_C := \{(x, y) \in X \times X | y - x \in C\}$$

is a partial ordering on X. If, in addition, C is pointed, then $\leq_C$ is antisymmetric.

Definition 4.1.6. *Let S be a nonempty subset of a partially ordered linear space with an ordering cone C.*

1. *An element $x^* \in S$ is called a **minimal element** of the set S if*

$$(\{x^*\} - C) \cap S \subseteq \{x^*\} + C.$$

2. *An element $x^* \in S$ is called a **maximal element** of the set S if*

$$(\{x^*\} + C) \cap S \subseteq \{x^*\} - C.$$

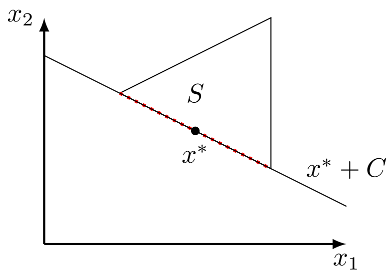

Figure 4.4: $x^* \in S$ is a minimal element of the set S, where C is a halfspace.

If, in addition, the ordering cone C has nonempty algebraic interior, we call an element $x^* \in S$

1. *a **weakly minimal element** of S if $(\{x^*\} - \mathrm{core}(C)) \cap S = \emptyset$;*

2. *a **weakly maximal element** of S if $(\{x^*\} + \mathrm{core}(C)) \cap S = \emptyset$.*

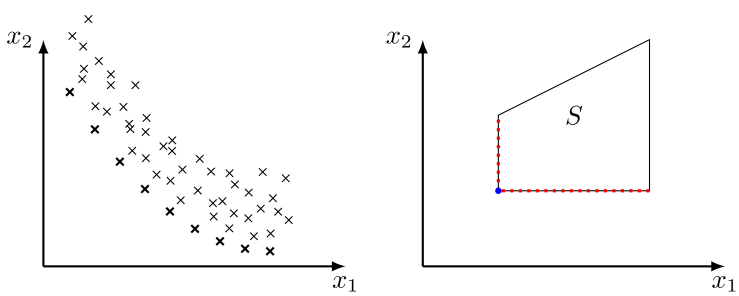

Figure 4.5: Here: $C = \mathbb{R}^2_+$. Left: The so-called **Pareto-frontier** of S, which consists of all minimal elements (here, S is a discrete set consisting of a finite number of vectors in $\mathbb{R}^2$). Right: The weakly minimal elements (red) and the minimal element (blue), where S is a continuous set.

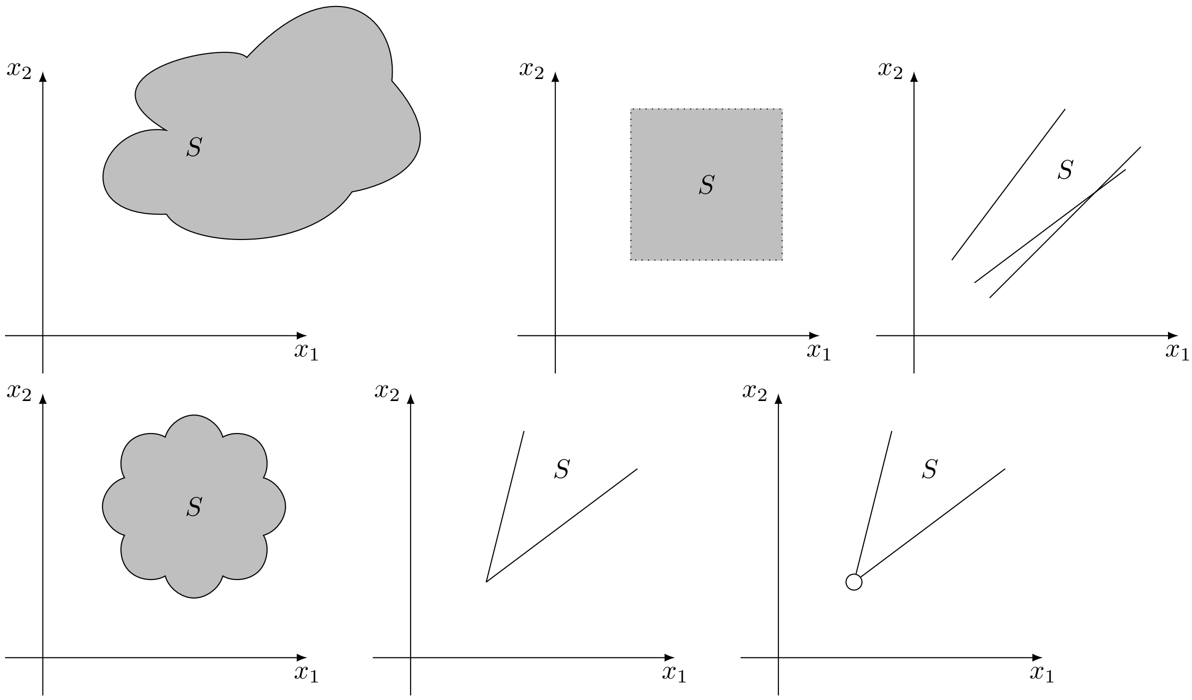

Figure 4.6:

Remark 4.1.6.1. *If the ordering cone C is pointed, then the above inclusions in Definition 4.1.3 can be replaced by the set equations*

$$(\{x^*\} - C) \cap S = \{x^*\}, \quad (\{x^*\} + C) \cap S = \{x^*\}.$$

Example 4.1.6.1 (Exercise)**.** *Sketch the minimal and weakly minimal elements in the following images of bicriteria optimization problems with $C = \mathbb{R}^2_+$ (see Figure 4.6).*

Lemma 4.1.7. *Let S be a nonempty subset of a partially ordered linear space X with an ordering cone $C \subseteq X$, $\text{core}(C) \neq \emptyset$ and $C \neq X$. Then every minimal (maximal, resp.) element of S is also a weakly minimal (maximal, resp.) element of S.*

Proof. Let x^* be a minimal element in S, hence,

$$(\{x^*\} - C) \cap S \subseteq \{x^*\} + C. \tag{4.1}$$

We have:

$$(\{x^*\} - \text{core}(C)) \cap S \subseteq (\{x^*\} - C) \cap S \overset{(4.1)}{\subseteq} \{x^*\} + C.$$

Hence, in other words, if $y \in S$ and $y \in \{x^*\} - \text{core}(C)$, then $y \in \{x^*\} + C$. This implies

$$y - x^* \in -\text{core}(C) \text{ and } y - x^* \in C.$$

However, we have $(-\operatorname{core}(C)) \cap C = \emptyset$, as $C \neq X$. Therefore,

$$(\{x^*\} - \operatorname{core}(C)) \cap S = \emptyset,$$

which means that x^* is a weakly minimal element in S. □

Of course, we could replace x^* in the above definition by $f(x^*)$ if we are minimizing a function $f : X \supseteq S \to Y$. However, here, we stick with finding minimal elements in S. The results to come are however easily adaptable to the case of minimizing a function f.

The following lemma indicates that the minimal elements of a set S and the minimal elements of the set $S + C$, where C denotes the ordering cone, are closely related.

Lemma 4.1.8. *Let S be a nonempty subset of a partially ordered linear space with a convex ordering cone C. Then every minimal element of the set S is also a minimal element of the set $S + C$.*

Proof. Take an arbitrary minimal element $x^* \in S$ of the set S, and choose any $x \in (\{x^*\} - C) \cap (S + C)$. Then there are elements $s \in S$ and $c \in C$ so that $x = s + c$. Consequently, we obtain $s = x - c \in \{x^*\} - C - C \subseteq \{x^*\} - C$ (as C is convex), and since x^* is a minimal element of the set S, we conclude $s \in \{x^*\} + C$. But then we get also $x \in \{x^*\} + C$. This completes the proof. □

Remark 4.1.8.1. *The converse direction of the above lemma holds true if C is pointed, see [Jahn, Lemma 4.7, p. 106].*

4.2 Jahn-Graef-Younes Methods

Here we investigate the special case that we want to determine the minimal elements of a set of finitely many points. In practice, such a set consists of many points so that it is not possible to use only the definition of minimality. Here we present a reduction approach which can be used for the elimination of non-minimal elements in such a set and for the determination of all minimal elements. In the following let S be a nonempty discrete subset of $\mathbb{R}^n$ being partially ordered in a natural way. In case we are given a continuous set, a discrete set can be obtained by discretization. Let S consist of many vectors. We are interested in the determination of all minimal elements of S.

For complexity reasons it does not make sense to determine all minimal elements using the definition. Therefore, one tries to reduce the set S, that is to eliminate those elements in S which cannot be minimal. Such a reduction of S can be carried out with the Graef- Younes method.

Algorithm 4.2.1. *(Graef-Younes-Method for sorting out non-minimal elements of a discrete set S)*

Input: $S := \{y^1, \dots, y^m\} \subset \mathbb{R}^n$, *ordering cone* $C \subset \mathbb{R}^n$
% Initialization
$\mathcal{T} := \{y^1\}$,
% Iteration loop
for $j = 2 : 1 : m$ ***do***
  if $(\{y^j\} - C) \cap \mathcal{T} \subseteq \{y^j\} + C$ ***then***
    $\mathcal{T} := \mathcal{T} \cup \{y^j\}$
  end if
end for
Output: $\mathcal{T}$

Theorem 4.2.2 (Theorem 12.18 in Jahn)**.** *1. Algorithm 4.2.1 is well-defined.*

2. Algorithm 4.2.1 generates a nonempty $\mathcal{T} \subseteq S$.

3. Every minimal element of S belongs to the set $\mathcal{T}$ generated by algorithm 4.2.1.

Proof. As the assertions under 1. and 2. are obvious, we prove only part 3. Let y^j be a minimal element of S. Suppose that $y^j \notin \mathcal{T}$. Obviously, $j \neq 1$. As y^j is a minimal element of S, we have

$$\left(\{y^j\} - C\right) \cap S \subseteq \{y^j\} + C.$$

Because $\mathcal{T} \subseteq S$, it holds that

$$(\{y^j\} - C) \cap \mathcal{T} \subseteq (\{y^j\} - C) \cap S \subseteq \{y^j\} + C.$$

This means that the if-condition in Algorithm 4.2.1 is fulfilled and y^j will be added to the set $\mathcal{T}$. This is a contradiction to our assumption. □

Algorithm 4.2.1 is a self learning method which becomes better and better step by step.

Next we discuss an extension of the Graef-Younes method. Algorithm 4.2.1 starts with a set S and generates a subset $\mathcal{T}$. If we apply Algorithm 4.2.1 to this set $\mathcal{T}$ with the modification that we check the elements of $\mathcal{T}$ from the right to the left, i.e. backwards with respect to the indices, we get the following method which generates all minimal elements of the set S under the assumption that the ordering cone C is pointed.

Algorithm 4.2.3. *(Jahn-Graef-Younes-Method with backward iteration for determining minimal elements of a discrete set S)*

Input: $S := \{y^1, \ldots, y^m\} \subset \mathbb{R}^n$, ***pointed*** *ordering cone* $C \subset \mathbb{R}^n$
% Initialization
$\mathcal{T} := \{y^1\}$
% Iteration loop: Forward iteration
for $j = 2 : 1 : m$ ***do***
  if $(\{y^j\} - C) \cap \mathcal{T} = \{y^j\}$ ***then***
    $\mathcal{T} := \mathcal{T} \cup \{y^j\}$
  end if
end for
$\{y^1, \ldots, y^p\} := \mathcal{T}$
$\mathcal{U} := \{y^p\}$
% Backward iteration
for $j = p - 1 : -1 : 1$ ***do***
  if $(\{y^j\} - C) \cap \mathcal{U} = \{y^j\}$ ***then***
    $\mathcal{U} := \mathcal{U} \cup \{y^j\}$
  end if
end for
Output: $\mathcal{U}$

We need the following notion in order to show that the above Jahn-Graef-Younes algorithm does in fact find all minimal elements of S.

Definition 4.2.4 (External Stability)**.** *Let the set of minimal elements of S w.r.t. C be denoted by S^C_{min}. If for all non-minimal elements $y \in S$ there exists a minimal element $\bar{y} \in S^C_{min}$ with $\bar{y} \in y - C$, then S^C_{min} is called **externally stable**.*

Remark 4.2.4.1. *It is well-known that every nonempty finite subset of a general preordered set is externally stable (see, e.g., Podinovskiĭ and Nogin: Pareto optimal solutions of multicriteria optimization problems (in Russian), Nauka, Moscow, 1982. p. 21).*

Theorem 4.2.5. *Let C be a pointed ordering cone. The set $\mathcal{U}$, generated by Algorithm 4.2.3, consists of exactly all minimal elements of S.*

Proof. We know from Theorem 4.2.2 that all minimal elements of S are included in $\mathcal{U}$. Now let us show that any element $y^j \in \mathcal{U}$ is minimal in S. After the forward iteration, we get

$$(\{y^j\} - C) \cap \mathcal{T} = \{y^j\},$$

where $\mathcal{T} := \{y^1, \ldots, y^{j-1}\}$. The backward iteration yields

$$(\{y^j\} - C) \cap \mathcal{U} = \{y^j\},$$

where $\mathcal{U} = \{y^p, \ldots, y^{j+1}\}$. This implies

$$(\{y^j\} - C) \cap \mathcal{V} = \{y^j\},$$

where $\mathcal{V} := \{y^1, \ldots, y^{j-1}, y^{j+1}, y^p\}$. Because $0 \in C$ (otherwise, C would not be a cone), we also have $(\{y^j\} - C) \cap \{y^j\} = \{y^j\} \subseteq \{y^j\} + C$ and thus, y^j is minimal in $\overline{\mathcal{V}} := \mathcal{V} \cup \{y^j\}$. Suppose that $\{y^j\}$ is not a minimal element in S. Then

$$(\{y^j\} - C) \cap S \neq \{y^j\},$$

which means that there exists $y^k \in S$ which is minimal in S with $y^k \in y^j - C$, hence $y^k \in (\{y^j\} - C) \cap S$, but $y^k \neq y^j$. As, due to Theorem 4.2.2, all minimal elements of S are included in $\mathcal{U}$, we have that $y^k \in \mathcal{U}$. Since $y^j \in \mathcal{U}$, we also have

$$(\{y^j\} - C) \cap \mathcal{U} = \{y^j\}.$$

As $y^k \in \mathcal{U}$ and $y^k \in y^j - C$, we obtain $y^k = y^j$, a contradiction. □

4.3 Scalarization

Here, we assume that X is a linear space (for example, $X = \mathbb{R}^n$) and that $S, C \subset X$.

Definition 4.3.1 (Dual Cone)**.** *The* ***dual cone*** *to a cone C is denoted by*

$$C^* := \{\ell \in X^* \mid \forall\, c \in C : \; \ell(c) \geq 0\}.$$

Theorem 4.3.2. *Let $x^* \in S$. If there exists an $\ell \in C^* \setminus \{0\}$ with*

$$\forall x \in S \setminus \{x^*\} : \quad \ell(x) < \ell(x^*),$$

then x^* is a minimal element in S.

Proof. Assume that x^* is not minimal. Then

$$(\{x^*\} - C) \cap S \not\subseteq \{x^*\} + C.$$

Hence, there exists an $\overline{x} \in S$ with $\overline{x} \in \{x^*\} - C$, but $\overline{x} \notin \{x^*\} + C$. The first assertion leads to $x^* - \overline{x} \in C$ and therefore $\ell(\overline{x}) \leq \ell(x^*)$ for any $\ell \in C^*$, while the second assertion ensures that $x^* \neq \overline{x}$. This contradicts the assumption. □

The inverse direction of the above assertion is in general **not** true.

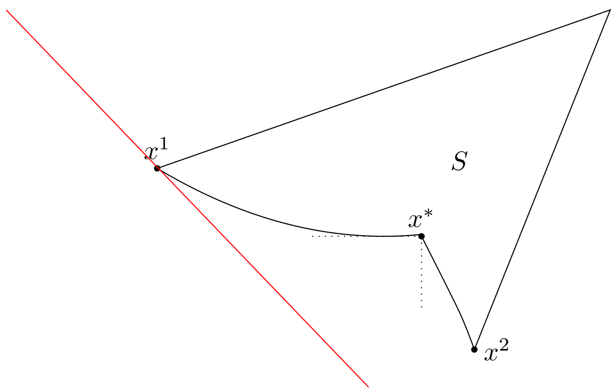

Figure 4.7: Here, $C = \mathbb{R}^2_+$. We have $(\{x^*\} - C) \cap S \subseteq \{x^*\} + C$; thus, x^* is minimal in S. However, x^* can not be found be means of linear scalarization. In fact, for this example, only the endpoints x^1, x^2 would be found by linear scalarization techniques.

Example 4.3.2.1. *In the finite-dimension case $X = \mathbb{R}^n$, the above result is called* ***weighted sum scalarization****, and the result reduces to: If there exists a weight vector $w \in \mathbb{R}^n_+ \setminus \{0\}$ with*

$$\forall x \in S \setminus \{x^*\} : \sum_{i=1}^{n} w_i x_i^* < \sum_{i=1}^{n} w_i x_i,$$

then x^* is a minimal element in S, where $C = \mathbb{R}^n_+$.

Theorem 4.3.3 (Basic Version of a Separation Theorem)**.** *Let S and T be nonempty and convex subsets of a real linear space X with* $\operatorname{core}(S) \neq \emptyset$. *Then* $\operatorname{cor}(S) \cap T = \emptyset$ *if and only if there exist a linear functional $\ell \in X^* \setminus \{0\}$ and a real number α with*

$$\forall s \in S, t \in T : \quad \ell(s) \leq \alpha \leq \ell(t)$$

and for all $s \in \operatorname{cor}(S)$:

$$\ell(s) < \alpha.$$

Theorem 4.3.4. *Let C be a convex cone, $S + C$ be closed and convex and* $\operatorname{core}(S + C) \neq \emptyset$. *If x^* is a minimal element in S, then there exists an $\ell \in C^* \setminus \{0\}$ s.t.*

$$\forall x \in S : \quad \ell(x^*) \leq \ell(x).$$

Proof. If x^* is a minimal element in S, then x^* is also minimal in $S + C$ due to Lemma 4.1.8. That is,

$$(\{x^*\} - C) \cap (S + C) \subseteq \{x^*\} + C.$$

Because $\{x^*\} - C$ and $S + C$ are convex, $\operatorname{core}(S + C) \neq \emptyset$ and $x^* \notin \operatorname{core}(S + C)$, by the separation theorem 4.3.3 there are a linear functional $\ell \in X^* \setminus \{0\}$ and a real number α with

$$\forall x \in S, c^1, c^2 \in C : \quad \ell(x^* - c^1) \leq \alpha \leq \ell(x + c^2).$$

Since C is a cone, we immediately obtain $\ell \in C^* \setminus \{0\}$ (otherwise, we would get a contradiction). By setting $c^1 = c^2 = 0$, we get

$$\forall x \in S : \quad \ell(x^*) \leq \ell(x).$$

$\square$