Norwegian University of Science and Technology
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TMA4180
Optimisation 1
Spring 2022

Solutions to exercise set 8

1 First of all we notice that since there are no inequalities of the form $x, y \geq 0$, we need to split them into their positive and negative parts as $x = x^+ - x^-$, $y = y^+ - y^-$, with $x^+ = \max\{0, x\}$ and $x^- = \max\{0, -x\}$. We then also introduce the slack variables s_1, s_2, z defined as

$$s_1 = u - y \geq 0, \quad s_2 = y - l \geq 0, \quad z = b_2 - A_2 x - B_2 y$$

We are finally in the position to introduce the solution vector

$$\bar{x} \ = [x^+, x^-, y^+, y^-, z, s_1, s_2]$$

that satisfies $\bar{x} \geq 0$. Remains to manage the equality constraints, this can be done defining the matrix and vectors $\bar{A}$ and $\bar{b}$ as follows

$$\bar{A}\bar{x} = \bar{b} \iff \begin{bmatrix} A_1 & -A_1 & 0 & 0 & 0 & 0 & 0 \\ A_2 & -A_2 & B_2 & -B_2 & I & 0 & 0 \\ 0 & 0 & I & -I & 0 & -I & 0 \\ 0 & 0 & I & -I & 0 & 0 & I \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \\ y^+ \\ y^- \\ z \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ l \\ u \end{bmatrix}$$

We conclude defining the vector $\bar{c} = -[c, -c, d, -d, 0, 0, 0]$ and the problem takes the desired form. We remark that the minus sign in front of the vector is because we have transformed the maximization problem into an equivalent minimization problem.

2 See the solution of Problem 4 in the exam of 2021 here `https://wiki.math.ntnu.no/_media/tma4180/2021v/completeexamsolutions.pdf`

3 We show that the two linear programs have the same KKT systems. We start studying the first linear program and call π the vector of Lagrangian multipliers associated to $Ax \geq b$, while let $s \geq 0$ be the one associated to $x \geq 0$. Thus, we get a Lagrangian function of the form

$$L_1(x, \pi, s) = c^T x - \pi^T (Ax - b) - s^T x.$$

This generates the following KKT conditions

$$\begin{aligned} A^T\pi + s &= c \\ Ax &\geq b \\ x &\geq 0 \\ \pi &\geq 0 \\ s &\geq 0 \\ \pi^T(Ax - b) &= 0 \\ s^T x &= 0. \end{aligned}$$

We do the same thing for the second problem. We have already seen in another exercise what are the KKT conditions for a maximization problem, but for simplicity we recover it transforming the system as follows

$$\max b^T\pi \iff \min -b^T\pi.$$

We let x be the Lagrange multiplier associated to $A^T\pi \leq c$ and y the one associated to $\pi \geq 0$. This gives the Lagrangian function

$$L_2(\pi, x, y) = -b^T\pi - x^T(c - A^T\pi) - y^T\pi$$

and thus the KKT conditions read

$$\begin{aligned} Ax - b &= y \\ A^T\pi &\leq c \\ \pi &\geq 0 \\ x &\geq 0 \\ y &\geq 0 \\ x^T(c - A^T\pi) &= 0 \\ y^T\pi &= 0. \end{aligned}$$

We can now conclude by noticing that $s = c - A^T\pi$ and $y = Ax - b$, since these two equalities allow to get the same KKT conditions.

$\boxed{4}$ We notice that the problem is already in standard form (LP3), where we identify A, b, c as

$$x = [x_1, x_2, x_3], \quad c = [5, 3, 4], \quad A = [1, 1, 1], \quad b = 1.$$

This implies that the dual is of the form

$$\max \lambda \quad \text{subject to} \begin{cases} \lambda \leq 5, \\ \lambda \leq 3, \\ \lambda \leq 4. \end{cases}$$

This can be easily solved by $\lambda^* = 3$. With this value of λ, introducing the slack variable $s = c - A^T\lambda$, we get that $s = [2, 0, 1]$. Let us now write the KKT conditions for the dual problem

$$\min -\lambda, \text{ subject to}, \lambda \leq 3, \lambda \leq 4, \lambda \leq 5.$$

The Lagrangian reads $\mathcal{L}(\lambda, x) = -\lambda - x^T(c - A^T\lambda)$ and the KKT conditions are

$$\begin{aligned} Ax &= b \\ A^T\lambda &\leq c \\ x &\geq 0 \\ x_i(c - A^T\lambda)_i &= 0,\ i = 1, 2, 3 \end{aligned}$$

as presented also in Formula 13.9, of N&W. This means that since $c - A^T\lambda = s = [2, 0, 1]$, for sure the solution x^* of the primal needs to have $x_1^* = x_3^* = 0$ in our case. Furthermore, we have that the value attained by the objective functions of the primal and dual problems, need to coincide at the points x^*, λ^* (i.e. strong duality holds). This gives

$$c^T x^* = 5x_1^* + 3x_2^* + 4x_3^* = 3x_2^* = \lambda^* = 3$$

and hence the optimal solution to the primal problem is $x^* = [0, 1, 0]$. One also can check that the constraint $Ax^* = b$ is satisfied.

5 See the solution of Problem 4 in the exam of 2015 here `https://wiki.math.ntnu.no/_media/tma4180/2015v/lf_summer15.pdf`