Norwegian University of Science and Technology Department of Mathematical Sciences	TMA4180 Optimisation 1 Spring 2022 **Solutions to exercise set 4**

$\boxed{1}$ As seen in the previous exercise sheet, a strongly-convex twice-differentiable function satisfies

$$p^T \nabla^2 f(x) p \geq \sigma \|p\|^2 \quad \forall p \in \mathbb{R}^n.$$

We have that by Taylor's theorem

$$\nabla f(x_{k+1}) = \nabla f(x_k) + \int_0^1 \left[\nabla^2 f(x_k + t\alpha_k p_k)\alpha_k p_k \right] dt.$$

This implies that

$$\begin{aligned} \alpha_k p_k^T y_k &= \alpha_k p_k^T [\nabla f_{k+1} - \nabla f_k] \\ &= \alpha_k^2 \int_0^1 \left[p_k^T \nabla^2 f(x_k + t\alpha_k p_k) p_k \right] dt \geq \\ \geq \sigma \alpha_k^2 \|p_k\|^2 > 0. \end{aligned}$$

We can hence conclude noticing that $\alpha_k p_k = s_k$ and hence $s_k^T y_k > 0$.

$\boxed{2}$ The second strong Wolfe condition implies that

$$\nabla f(x_k + \alpha_k p_k)^T p_k \geq -c_2 |\nabla f(x_k)^T p_k)| = c_2 \nabla f(x_k)^T p_k$$

if we recall that p_k is a descent direction. Thus

$$\begin{aligned} s_k^T y_k = \alpha_k p_k^T (\nabla f_{k+1} - \nabla f_k) &= \alpha_k p_k^T (\nabla f(x_k + \alpha_k p_k) - \nabla f(x_k)) \geq \\ \geq \alpha_k p_k^T (c_2 \nabla f(x_k) - \nabla f(x_k)) &= \alpha_k (c_2 - 1) p_k^T \nabla f(x_k) > 0 \end{aligned}$$

since $c_2 - 1 < 0$ and so is $p_k^T \nabla f(x_k)$. This means that the curvature condition holds.

$\boxed{3}$ When dealing with one-dimensional functions, all gradients and hessians become just scalar function. Recall that we always set $\alpha_k = 1$ in this exercise. The iteration of BFGS and of DFP method hence write

$$x_{k+1} = x_k - H_k^{BFGS} f'(x_k),$$

$$x_{k+1} = x_k - H_k^{DFP} f'(x_k).$$

For BFGS method we have that called $\rho_k = 1/(s_k y_k)$,

$$H_{k+1}^{BFGS} = (1 - \rho_k s_k y_k) H_k^{BFGS} (1 - \rho_k y_k s_k) + \rho_k s_k^2 = \frac{s_k}{y_k} = \frac{x_{k+1} - x_k}{f'(x_{k+1}) - f'(x_k)}$$

since $1 - \rho_k s_k y_k = 0$. This gives exactly the update rule of the secant method. For DFP method we have (see equation 6.15 Nocedal and Wright)

$$H_{k+1}^{DFP} = H_k^{DFP} - \frac{H_k^{DFP} y_k y_k H_k^{DFP}}{y_k H_k^{DFP} y_k} + \frac{s_k^2}{s_k y_k} = \frac{s_k}{y_k} = \frac{x_{k+1} - x_k}{f'(x_{k+1}) - f'(x_k)}$$

that gives the same expression of the secant method.

$\boxed{4}$ You can find an implementation to solve this exercise here `https://github.com/davidemurari/TMA4180_2022/tree/main/ExSheet4`