Now: let $\mathcal{B}_= := \{ x \in \mathbb{R}^n \mid Ax = b, x \geq 0 \}$

Theorem: $x \in \mathcal{B}_=$ is a vertex of $\mathcal{B}_= \iff$
x can be written as intersecting point of n linear independent hyperplanes.

Proof: "$\Leftarrow$" let $x^0 = (x_1^0, \dots, x_n^0) \in \mathcal{B}_=$
be the point of intersection of n linear independent hyperplanes.
Suppose that x^0 is not a vertex.
Then there exist $x^1, x^2 \in \mathcal{B}_=$
with $x^0 = \lambda x^1 + (1-\lambda) x^2$,
$\lambda \in [0,1]$. We assume that the first r coordinates in x^0 are nonzero, and that $a^1, \dots, a^r$ are linearly independent.
For $k > r$, we have $x_k = 0$
and therefore also $x_k^1 = x_k^2 = 0$.
From
$$\sum_{k=1}^{r} a^k (\lambda x^1 + (1-\lambda) x^2) = b,$$
we get $\sum_{k=1}^{r} a^k (x^1 - x^2) = 0$
(because $x^1, x^2 \in \mathcal{B}_=$).

Because the hyperplanes are linearly independent, we get $x^1 = x^2$, a contradiction.

" $\Longrightarrow$ " Let $x^0 \in \mathcal{B}_=$ be a vertex of $\mathcal{B}_=$. We assume that the components of x^0 are numbered s.t.

$$x_1^0, \dots, x_r^0 > 0, \quad x_{r+1}^0 = \dots = x_n^0 = 0$$

$(0 < r \leq n)$. Thus we have

$$\sum_{k=1}^{r} a^{(k)} x_k^0 = b.$$

Assume, by contradiction, that $a^{(1)}, \dots, a^{(r)}$ are linearly dependent.

Let $\sum_{k=1}^{r} a^{(k)} d_k = 0$. Then, for some k, $d_k \neq 0$.

Then we have for some sufficiently small $\delta > 0$:

$$\sum_{k=1}^{r} a^{(k)} (\underbrace{x_k \pm \delta d_k}_{\geq 0}) = b.$$

This means:

$$x^1 = \sum_{k=1}^{r} a^{(k)} (x_k + \delta d_k) = b$$

$$x^2 = \sum a^{(k)} (x_k - \delta d_k) = b$$

and $x^0 = \frac{1}{2}(x^1 + x^2)$ and this contradicts the assumption that x^0

is a vertex. $\blacksquare$

<u>Theorem</u>: There are at most finitely many vertices.

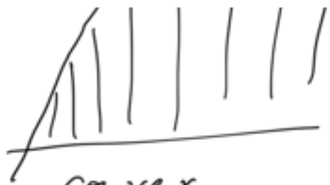
<u>Proof</u>: There are only finitely many systems of m linear independent columns (vertices can only exist if $n \le m$, i.e., we have more constraints than unknowns)
$\Rightarrow$ There are at most finitely many vertices. $\blacksquare$

Because A has at most m linear independent columns, we get:

<u>Theorem</u>: The vertex x has at most m positive coordinates, the remaining ones are zero.

<u>Definition</u>: The set of points satisfying a system of linear inequalities is called a <u>convex polyhedron</u>. If a convex polyhedron is bounded, it is called a <u>polytope</u>.

<u>Remark</u>: $\mathcal{B} = \{x \in \mathbb{R}^n \mid Ax \le b\}$ is a convex polyhedron.



convex polyhedron

polytope

<u>Theorem</u>: If $\mathcal{B}$ is a polytope, then every point of $\mathcal{B}$ is a convex combination of the finitely many vertices in $\mathcal{B}$.

<u>Theorem</u>: A concave function attains its global minimize on a polytope $\mathcal{B}$ in a vertex of $\mathcal{B}$.

<u>Proof</u>: Let $x^1, \dots, x^k$ denote the vertices of $\mathcal{B}$. Then, every $x \in \mathcal{B}$ can be characterized by

$$x = \sum_{i=1}^{k} \lambda_i x^i \; , \quad \sum_{i=1}^{k} \lambda_i = 1 \, ,$$

$$\lambda_i \geq 0 \, , \quad \forall i = 1, \dots, k .$$

That means that every $x \in \mathcal{B}$ can be written as a convex combination of the finitely many vertices of $\mathcal{B}$.

There exists $\min\limits_{i=1,\dots,k} f(x^i) =: \underline{f(x^j)}$.

Then it holds for arbitrary $x \in \mathcal{B}$:

$$f(x) = f\Big(\sum_{i=1}^{k} \lambda_i x^i \Big) \underset{\substack{\uparrow \\ f \text{ concave}}}{\geq} \sum_{i=1}^{k} \lambda_i f(x^i)$$

$$\geq \underbrace{\sum_{i=1}^{n} \lambda_i}_{1} f(x^j)$$

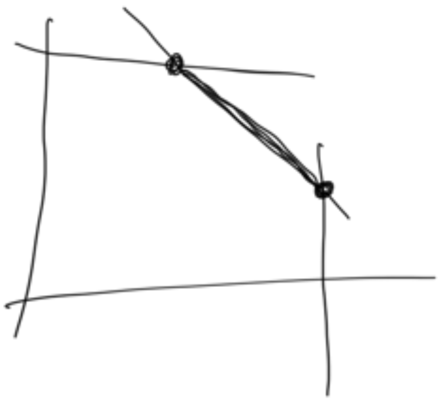
$$= f(x^j)$$

$\Rightarrow \forall x \in \mathcal{B}: \quad f(x) \geq f(x^j),$

x^j is a vertex.

$\Rightarrow$ The minimize of f on $\mathcal{B}$ exists and is attained in the vertex x^j. $\blacksquare$

<u>Conclusion</u>: When looking for a global minimizer, it suffices to check the vertices of $\mathcal{B}$.



We use that the objective function is <u>linear</u>:

- <u>convex</u>: Every local minimize is a global minimize
- <u>concave</u>: The global minimize is attained in a vertex.

$\Rightarrow$ We need a method that systematically evaluates the function values at the vertices.

$\Rightarrow$ simplex algorithm.

Duality for linear programs

Consider the following primal program:

$$(LP3) \qquad \min \ c^T x =: Q(x)$$
$$\text{s.t.} \quad Ax \leq b$$
$$x \geq 0$$

We construct a problem which is "dual" to (LP3):

$$(LP3^D) \qquad \max \ b^T u =: G(u)$$
$$\text{s.t.} \quad A^T u \leq c$$

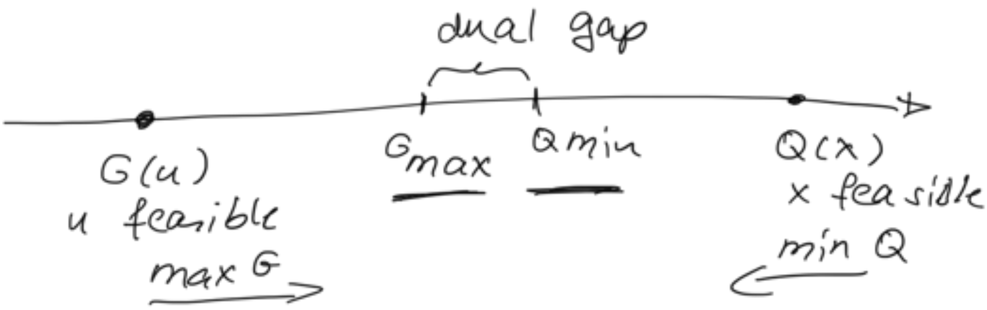
$$(A^T \in \mathbb{R}^{(n,m)}, \ \text{rank}\,(A^T) = m)$$

Theorem (Weak Duality): Let x be feasible for (LP3) and u be feasible for $(LP3^D)$. Then

$$G(u) \leq Q(x).$$

<u>Proof</u>: For arbitrary feasible elements x, u, it holds:

$$G(u) = u^T b \underset{\substack{\uparrow \\ Ax = b}}{=} u^T(Ax) \underset{\substack{\uparrow \\ A^T u \leq c}}{\leq} c^T x = Q(x).$$



$\Rightarrow$

- Every objective function value $G(u)$ of a feasible u is a lower bound for arbitrary function values of Q.
- Every objective function value $Q(x)$ of a feasible x is an upper bound for arbitrary function values of G.

<u>Theorem</u>: If (LP3) has a ~~minimizer~~ solution, then $(LP3^D)$ has a ~~minimizer~~ solution, and vice versa. In that case, we have

$$Q(x^0) = G(u^0) = G_{max}$$

$$Q_{min} = Q(x$$

<u>Existence Theorem</u>: If (LP3) and (LP3^D) have feasible points, then both problem have a solution ~~minimize~~ and their objective values are equal.

We have:

		(LP3)	
		$\exists$ feasible point	$\nexists$ feasible point
(LP3^D)	$\exists$ feasible point	$\min Q = \cancel{\min}\ \max G$	$G \to +\infty$
	$\nexists$ feas. point	$Q \to -\infty$	—