TMA4180 Optimization: Minimizers

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(Strict) Global Minimizers

We will discuss the existence of solutions of minimization problems of the form

$$\min_{x \in \Omega} f(x),$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a real valued function (the cost function or objective function) and $\Omega \subset \mathbb{R}^n$ is some set (the feasible set).

Global Minimizer
A point $x^* \in \Omega$ is called a global minimizer (or global minimum) of the optimization problem $\min_{x \in \Omega} f(x)$, if

$$f(x^*) \leq f(x)$$

for all $x \in \Omega$.

Strict Global Minimizer
The point $x^*$ is strict global minimizer, if $f(x^*) < f(x)$ for all $x \in \Omega$, $x \neq x^*$. 
(Strict) Global Minimizer

(Strict) Global minimizers need not exist, as one can see in the following examples (see also the figure below):

- Minimize the function $f(x) = e^{-x^2}$ for $x \in \mathbb{R}$ (that is, $\Omega = \mathbb{R}$).

The function $f(x) = e^{-x^2}$ obviously does not attain its minimum, because of the drop-off of the function values near infinity.
Minimize the function \( f(x) = x \) for \( x > 0 \) and \( f(x) = x^2 + 1 \) for \( x < 0 \).

The existence of a global minimizer of the function \( f \) defined by \( f(x) = x^2 + 1 \) for \( x < 0 \) and \( f(x) = x \) for \( x > 0 \) depends on its value at 0. If \( f(0) \leq 0 \), the point \( x = 0 \) is the strict global minimizer. If, however, \( f(0) > 0 \), there does not exist a global minimizer.
Remark
Global minimizers need not be unique. For example, for the function $f(x) = c$ with a constant $c \in \mathbb{R}$, every point $x \in \mathbb{R}$ is a global minimizer (but there are no strict global minimizers).
Local Minimizer

A point \( x^* \in \Omega \) is called a *local minimizer* of the optimization problem \( \min_{x \in \Omega} f(x) \), if there exists \( \epsilon > 0 \) such that \( f(x^*) \leq f(x) \) whenever \( x \in \Omega \) satisfies \( \|x - x^*\| \leq \epsilon \).
Strict Local Minimizer

A point $x^*$ is called a strict local minimizer of $\min_{x \in \Omega} f(x)$, if there exists $\epsilon > 0$ such that

$$f(x^*) < f(x)$$

whenever $x \in \Omega$, $x \neq x^*$ satisfies $\|x - x^*\| \leq \epsilon$. 
Weierstrass Extreme Value Theorem
Lower Limit of a Sequence of Real Numbers

Definition
The lower limit of a sequence of real numbers \((z_k)_{k \in \mathbb{N}}\) is defined as

\[
\liminf_{k \to \infty} z_k := \lim_{k \to \infty} \inf_{\ell \geq k} z_\ell.
\]
Lower Semi-Continuity

Definition
A function $f : \mathbb{R}^n \to \mathbb{R}$ is called lower semi-continuous, if for every $x \in \mathbb{R}^n$ and every sequence $(x_k)_{k \in \mathbb{N}}$ converging to $x$ we have

$$f(x) \leq \liminf_{k \to \infty} f(x_k).$$
Remark
An alternative (equivalent) definition of lower semi-continuity is the following: A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is lower semi-continuous, if the *lower level set*

$$\Omega_\alpha := \{ x \in \mathbb{R}^n : f(x) \leq \alpha \}$$

is closed for every $\alpha \in \mathbb{R}$. 


Existence Result for Optimization Problems

The following proposition is an extension of the famous Weierstrass extreme value theorem:

**Proposition**

Assume that \( f : \mathbb{R}^n \to \mathbb{R} \) is lower semi-continuous and that \( \Omega \subset \mathbb{R}^n \) is compact. Then the optimization problem

\[
\min_{x \in \Omega} f(x)
\]

at least one global minimizer.
Existence Result for Optimization Problems
Existence Result for Optimization Problems

Definition
A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called coercive, if we have for every sequence $(x_k)_{k \in \mathbb{N}}$ with $\|x_k\| \rightarrow \infty$ that $f(x_k) \rightarrow \infty$.

Theorem
Assume that the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is lower semi-continuous and coercive. Then the optimization problem $\min_{x \in \mathbb{R}^n} f(x)$ admits at least one global minimizer.
Existence Result for Optimization Problems