TMA4180 Optimization: Minimizers

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(Strict) Global Minimizers

We will discuss the existence of solutions of minimization problems of the form

$$\min_{x \in \Omega} f(x),$$

where $f \colon \mathbb{R}^n \to \mathbb{R}$ is a real valued function (the *cost function* or *objective function*) and $\Omega \subset \mathbb{R}^n$ is some set (the *feasible set*).

Global Minimizer

A point $x^* \in \Omega$ is called a *global minimizer* (or *global minimum*) of the optimization problem $\min_{x \in \Omega} f(x)$, if

$$f(x^*) \leq f(x)$$

for all $x \in \Omega$.

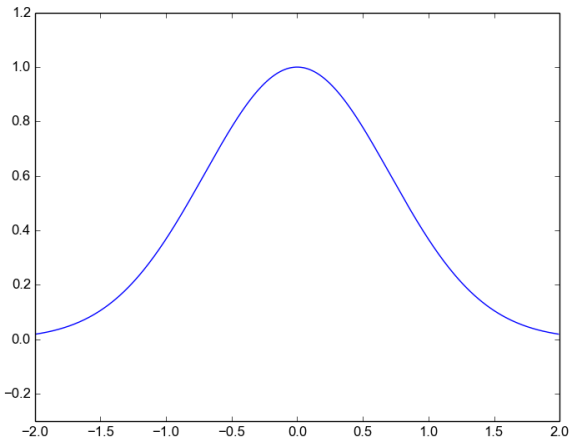
Strict Global Minimizer

The point x^* is *strict global minimizer*, if $f(x^*) < f(x)$ for all $x \in \Omega$, $x \neq x^*$.

(Strict) Global Minimizer

(Strict) Global minimizers need not exist, as one can see in the following examples (see also the figure below):

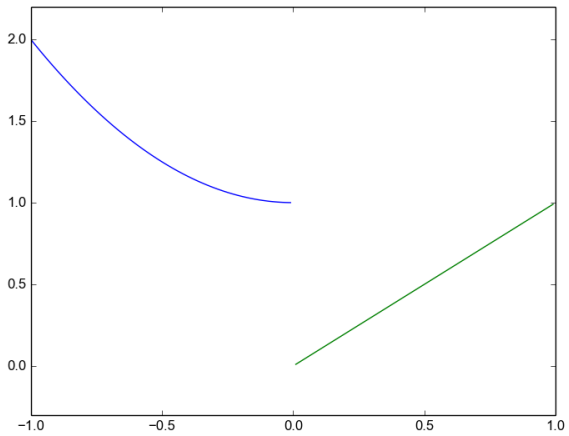
- Minimize the function $f(x) = e^{-x^2}$ for $x \in \mathbb{R}$ (that is, $\Omega = \mathbb{R}$).



The function $f(x) = e^{-x^2}$ obviously does not attain its minimum, because of the drop-off of the function values near infinity.

(Strict) Global Minimizer

- Minimize the function $f(x) = x$ for $x > 0$ and $f(x) = x^2 + 1$ for $x < 0$.



The existence of a global minimizer of the function f defined by $f(x) = x^2 + 1$ for $x < 0$ and $f(x) = x$ for $x > 0$ depends on its value at 0. If $f(0) \leq 0$, the point $x = 0$ is the strict global minimizer. If, however, $f(0) > 0$, there does not exist a global minimizer.

(Strict) Global Minimizer

Remark

Global minimizers need not be unique. For example, for the function $f(x) = c$ with a constant $c \in \mathbb{R}$, every point $x \in \mathbb{R}$ is a global minimizer (but there are no strict global minimizers).

Local Minimizer

Local Minimizer

A point $x^* \in \Omega$ is called a *local minimizer* of the optimization problem $\min_{x \in \Omega} f(x)$, if there exists $\epsilon > 0$ such that $f(x^*) \leq f(x)$ whenever $x \in \Omega$ satisfies $\|x - x^*\| \leq \epsilon$.

Strict Local Minimizer

Strict Local Minimizer

A point x^* is called a *strict local minimizer* of $\min_{x \in \Omega} f(x)$, if there exists $\epsilon > 0$ such that

$$f(x^*) < f(x)$$

whenever $x \in \Omega$, $x \neq x^*$ satisfies $\|x - x^*\| \leq \epsilon$.

Weierstrass Extreme Value Theorem

Lower Limit of a Sequence of Real Numbers

Definition

The lower limit of a sequence of real numbers $(z_k)_{k\in\mathbb{N}}$ is defined as

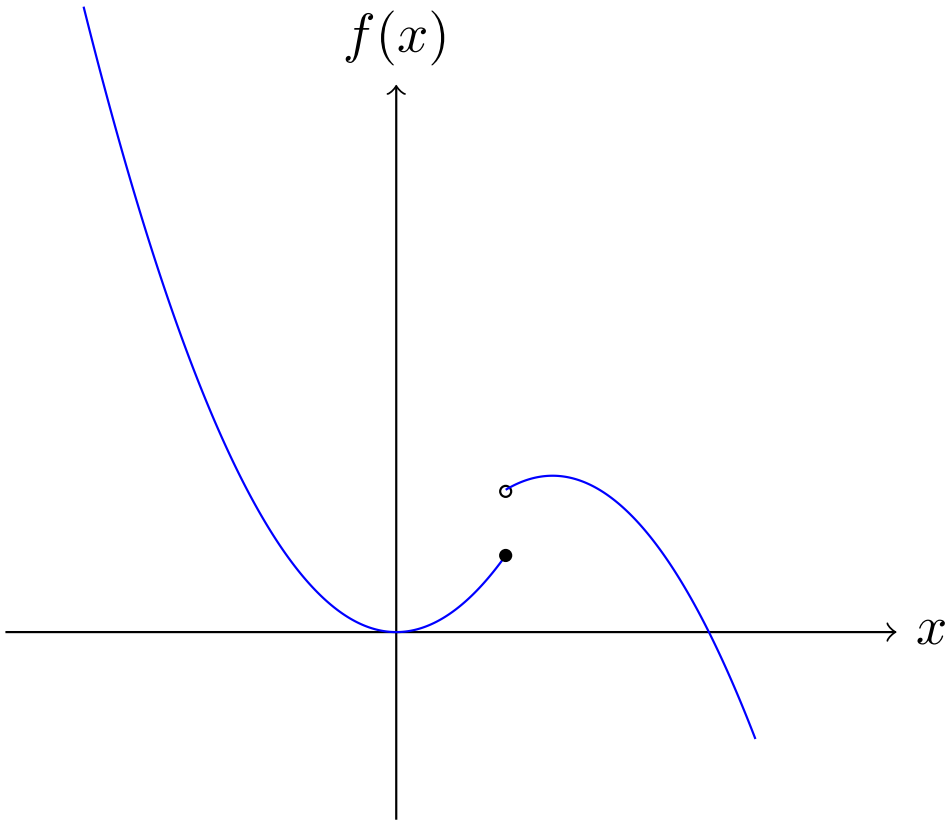
$$\liminf_{k\to\infty} z_k := \lim_{k\to\infty} \inf_{\ell\geq k} z_\ell.$$

Lower Semi-Continuity

Definition

A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called *lower semi-continuous*, if for every $x \in \mathbb{R}^n$ and every sequence $(x_k)_{k\in\mathbb{N}}$ converging to x we have

$$f(x) \leq \liminf_{k\to\infty} f(x_k).$$



Lower Semi-Continuity

Remark

An alternative (equivalent) definition of lower semi-continuity is the following: A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is lower semi-continuous, if the *lower level set*

$$\Omega_\alpha := \{x \in \mathbb{R}^n : f(x) \leq \alpha\}$$

is closed for every $\alpha \in \mathbb{R}$.

Existence Result for Optimization Problems

The following proposition is an extension of the famous Weierstrass extreme value theorem:

Proposition

Assume that $f\colon \mathbb{R}^n \to \mathbb{R}$ is lower semi-continuous and that $\Omega \subset \mathbb{R}^n$ is compact. Then the optimization problem

$$\min_{x \in \Omega} f(x)$$

at least one global minimizer.

Existence Result for Optimization Problems

Existence Result for Optimization Problems

Definition

A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called coercive, if we have for every sequence $(x_k)_{k\in\mathbb{N}}$ with $\|x_k\| \to \infty$ that $f(x_k) \to \infty$.

Theorem

Assume that the function $f\colon \mathbb{R}^n \to \mathbb{R}$ is lower semi-continuous and coercive. Then the optimization problem $\min_{x\in\mathbb{R}^n} f(x)$ admits at least one global minimizer.

Existence Result for Optimization Problems