TMA4180 Optimization: Introduction

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Mathematical Formulation of Optimization

Mathematically speaking, optimization is the minimization or maximization of a function subject to constraints on its variables.We use the following notation:

- x is the vector of variables, also called unknowns or parameters;
- f is the objective function, a (scalar) function of x that we want to maximize or minimize;
- c_i are constraint functions, which are scalar functions of x that define certain equations and inequalities that the unknown vector x must satisfy
- $\mathcal{E}$ and $\mathcal{I}$ are sets of indices for equality and inequality constraints, resp.

Using this notation, the optimization problem can be written as follows:

$$\begin{aligned} &\min_{x\in\mathbb{R}^n} f(x) \\ \text{subject to } &c_i(x) = 0, \quad i \in \mathcal{E}. \\ &c_i(x) \geq 0, \quad i \in \mathcal{I}. \end{aligned} \tag{P}$$

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We define: $\Omega := \{x \in \mathbb{R}^n | c_i(x) = 0, \quad i \in \mathcal{E}, c_i(x) \geq 0, \quad i \in \mathcal{I}\}$.

Example

Continuous vs. Discrete Optimization

Continuous Optimization

The solution variable x can be real-valued.

Discrete Optimization

The solution variable x is drawn from a discrete (but often very large) set, for example:

- $x \in \mathbb{Z}$ ((mixed) integer programming problems)
- $x \in \{0, 1\}$ (binary variables)

Constrained vs. Unconstrained Optimization

Constrained Optimization

The solution variable x is required to satisfy constraints (i.e., $\Omega \neq \mathbb{R}^n$).

Unconstrained Optimization

The solution variable x is **not** required to satisfy any constraints ($\mathcal{E} = \mathcal{I} = \emptyset$, or $\Omega = \mathbb{R}^n$). Unconstrained problems arise also as reformulations of constrained optimization problems, in which the constraints are replaced by penalization terms added to objective function that have the effect of discouraging constraint violations

Linear vs. Nonlinear Optimization

Linear Optimization

The optimization problem contains a **linear objective function** and only **linear constraints**.

Nonlinear Optimization

The solution variable x contains a **nonlinear objective function** and/or **nonlinear constraints**.

Stochastic vs. Deterministic Optimization

Stochastic Optimization

Sometimes, the model cannot be fully specified because it depends on quantities that are unknown at the time of formulation $\longrightarrow$ incorporate additional knowledge about these uncertain quantities into the model (for example, a number of possible scenarios for the uncertain demand, along with estimates of the probabilities of each scenario)

- Stochastic optimization: Algorithms use these quantifications of the uncertainty to produce solutions that optimize the expected performance of the model
- Chance-constrained optimization: Ensure that the variables satisfy the given constraints to some specified probability.
- Robust optimization: Certain constraints are required to hold for all possible values of the uncertain data.

Deterministic Optimization

The model is completely known.

Convexity

Convex Set

A set $S \subseteq \mathbb{R}^n$ is **convex** if for all $x, y \in S$, $\lambda \in [0, 1]$

$$\lambda x + (1 - \lambda)y \in S.$$

In other words: The line segment connecting x and y is also in S.

Remark

A point of the form $\lambda x + (1 - \lambda)y$, $\lambda \in [0, 1]$, is called a convex combination of x and y.

Convex Set – Nonconvex Set

Convexity

Convex Function

A function $f : S \to \mathbb{R}$ is **convex** if its domain S is a convex set and if for any two points $x, y \in S$ and $\lambda \in [0, 1]$ the following property is satisfied:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

In words: Take any two points x, y; The function value evaluated at any convex combination should not be larger than the same convex combination of $f(x)$ and $f(y)$. Geometrically, the line segment connecting $(x, f(x))$ to $(y, f(y))$ does not sit below the graph of f.

Concavity

Concave Function

A function $f : S \to \mathbb{R}$ is **concave** if its domain S is a convex set and if for any two points $x, y \in S$ and $\lambda \in [0, 1]$ the following property is satisfied:

$$f(\lambda x + (1 - \lambda)y) \boxed{\geq} \lambda f(x) + (1 - \lambda) f(y).$$

Remark

f is concave if and only if $-f$ is convex.

Convexity

Convex Programming

The term **convex programming** is used to describe a special case of the general constrained optimization problem (P) in which

- the objective function is convex,
- the equality constraint functions $c_i, i \in \mathcal{E}$, are linear, and
- the inequality constraint functions $c_i, i \in \mathcal{I}$, are concave.

Optimization Algorithms

Iterative Algorithms

Iterative Algorithm begin with an initial guess of the variable x and generate a sequence of improved estimates (called "‘iterates"’) until they terminate.

Properties of Algorithms

Good algorithms should possess the following properties:

- Robustness. They should perform well on a wide variety of problems in their class, for all reasonable values of the starting point.
- Efficiency. They should not require excessive computer time or storage.
- Accuracy. They should be able to identify a solution with precision, without being overly sensitive to errors in the data or to the arithmetic rounding errors that occur when the algorithm is implemented on a computer.

Notes

- Calculus of variations
- Optimization = Mathematical Programming
- Another interesting field: Modeling