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Assume that $\Omega \subset \mathbb{R}^n$ is a closed and convex set. Recall that we can define in this case for $x \in \Omega$ the normal cone $N_\Omega(x)$ as

$$N_\Omega(x) = \big\{q \in \mathbb{R}^n : q^T(\hat{x} - x) \le 0 \text{ for all } \hat{x} \in \Omega\big\},$$

and the tangent cone $T_\Omega(x)$ as

$$T_\Omega(x) = \big\{p \in \mathbb{R}^n : p^T q \le 0 \text{ for all } q \in N_\Omega(x)\big\}.$$

1 Assume that $\Omega \subset \mathbb{R}^n$ is non-empty, convex, and closed, and that $x \in \Omega$. Show that $N_\Omega(x)$ and $T_\Omega(x)$ are non-empty, closed, and convex cones.[1]

2 Consider the sets

$$\Omega_1 := \big\{x \in \mathbb{R}^n : \|x\|_\infty \le 1\big\}$$

and

$$\Omega_2 := \big\{x \in \mathbb{R}^n : \|x\|_2 \le 1\big\}.$$

a) Show that Ω_1 and Ω_2 are non-empty, convex, and closed.

b) In dimension $n = 2$, determine the normal and tangent cones to the sets Ω_1 and Ω_2 at the point $x = (1, 0)$. In addition, determine the normal and tangent cones to Ω_1 at the point $x = (1, 1)$.

Hint: This exercise will probably become easier if you start with drawing a sketch.

c) Consider the projection problems

$$\pi_{\Omega_i}(z) = \arg\min_{x \in \Omega_i} \frac{1}{2}\|x - z\|_2^2.$$

Use the variational characterisation of the projection ($(\pi_{\Omega_i}(z) - z)^T(x - \pi_{\Omega_i}(z)) \ge 0$ for all $x \in \Omega_i$) to verify the formulas

$$\pi_{\Omega_1}(z) = \big(\max\{-1, \min\{1, z_i\}\}\big)_{i=1,\ldots,n}$$

and

$$\pi_{\Omega_2}(z) = \frac{z}{\max\{1, \|z\|_2\}}.$$

[1] A subset $C \subset \mathbb{R}^n$ is called a cone, if, whenever $p \in C$, then also $\lambda p \in C$ for all $\lambda > 0$.

d) Consider now the case $n = 2$ and let

$$f(x) = x_1^2 + (x_2 + 2)^2.$$

Find the global solution of the problem

$$\min_{x \in \Omega_1} f(x)$$

(you can do this graphically, if you want) and perform one step of the gradient projection method

$$x^{(k+1)} \leftarrow \pi_{\Omega_1}(x^{(k)} - \alpha \nabla f(x^{(k)}))$$

with step length $\alpha = 1/2$ starting at $x^{(0)} = (1, 1)$.

3 Assume that $\Omega \subset \mathbb{R}^n$ is non-empty, convex and closed. Show that the projection mapping $\pi_\Omega \colon \mathbb{R}^d \to \Omega$ is a non-expansive map in the sense that

$$\|\pi_\Omega(x) - \pi_\Omega(y)\|_2 \leq \|x - y\|_2$$

for all x, $y \in \mathbb{R}^n$.

Hint: Show first that

$$(\pi_\Omega(x) - x)^T(\pi_\Omega(y) - \pi_\Omega(x)) \geq 0 \qquad \textit{and} \qquad (\pi_\Omega(y) - y)^T(\pi_\Omega(x) - \pi_\Omega(y)) \geq 0.$$

Then consider the function $g \colon \mathbb{R} \to \mathbb{R}$,

$$g(\lambda) := \|\lambda(x - y) + (1 - \lambda)(\pi_\Omega(x) - \pi_\Omega(y))\|_2^2.$$

Show that $g'(0) \geq 0$ *and deduce from the fact that* g *is quadratic that* $g(1) \geq g(0)$.

4 Let $A \in \mathbb{R}^{m \times n}$ with $m \geq n$ have full rank, let $b \in \mathbb{R}^m$, and let $\Omega \subset \mathbb{R}^n$ be non-empty, convex, and closed. Consider the restricted least squares problem

$$\min_{x \in \Omega} f(x) \qquad \text{where } f(x) = \frac{1}{2}\|Ax - b\|_2^2 \tag{1}$$

and the gradient projection algorithm

$$x^{(k+1)} \leftarrow \pi_\Omega\big(x^{(k)} - \alpha \nabla f(x^{(k)})\big).$$

Show that this algorithm converges to the unique solution of (1) provided that $0 < \alpha < 2/\sigma_{\max}^2$, where $\sigma_{\max}$ denotes the largest singular value of A.

Hint: Show that the gradient descent step $x \mapsto x - \alpha \nabla f(x)$ *is a contraction on* $\mathbb{R}^n$*, and then use the result of the previous exercise.*