Norwegian University of Science and Technology
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TMA4180 Optimisation I Spring 2020

Exercise set 2

1 Consider the function

$$f(x,y,z) = 2x^2 + xy + y^2 + yz + z^2 - 6x - 7y - 8z + 9.$$

a) Find all points $(x,y,z) \in \mathbb{R}^3$ satisfying the first order necessary conditions for this problem (critical points).

b) Assess whether the critical points satisfy the second order necessary/sufficient optimality conditions.

c) Verify that the function f is convex, and conclude that all critical points of f are global minima.

d) Let now $(\hat{x}, \hat{y}, \hat{z}) = (0,0,0)$ and let $p = (1,2,0)$. Verify that p is a descent direction for f at $(0,0,0)$. Find the range of step lengths $\alpha > 0$ that satisfy the Armijo condition for steps from $(0,0,0)$ in direction p with $c_1 = 4/5$.

e) With the notation/assumptions of **d)**, determine the step length you would obtain by using a backtracking line search with the Armijo condition (with parameter $c_1 = 4/5$) with an initial step length $\hat{\alpha} = 1$, and a step length reduction parameter of $\rho = 1/4$.

2 Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) = \log(e^x + e^y)$$

is convex.

3 *(See N&W, Exercise 2.8)* Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function. Show that the set of minimisers of f is convex (possibly empty, though).

4 Show that a strictly convex function $f : \mathbb{R}^n \to \mathbb{R}$ has at most one global minimiser. In addition, find a strictly convex function that has no global minimiser at all.

5 Assume that f is a continuously differentiable function satisfying

$$\lim_{\|x\|\to\infty} \frac{f(x)}{\|x\|} = +\infty.$$

Show that the equation

$$\nabla f(x) = u$$

has a solution for every $u \in \mathbb{R}^n$.

Hint: Consider global minima of the function $f_u(x) := f(x) - u^T x$.

6 Implement both the gradient descent method and Newton's method with a line search satisfying the Wolfe conditions (you may want to use a bisection algorithm for the implementation of these conditions)

Apply your method to the minimisation of the Rosenbrock function

$$f(x, y) := 100(y - x^2)^2 + (1 - x)^2.$$

The Newton direction is not necessarily a descent direction for this function, as f is not convex, and thus it might be necessary to modify the search directions in the Newton method. Do this by switching to the negative gradient direction, whenever the inequality

$$-\nabla f(x_k)^T p_k^{\text{Newton}} \le \varepsilon \|\nabla f(x_k)\| \|p_k^{\text{Newton}}\|$$

holds (here, $\varepsilon > 0$ is some fixed, small parameter).