Norwegian University of Science and Technology
Department of Mathematical Sciences

Exercise set 1

1 *(Properties of the lower limit)*

a) Consider the sequence $y_k = (k \bmod 3) - 1/k$, where $k \bmod 3 \in \{0, 1, 2\}$ is the remainder of integer division of k by 3. Does this sequence converge? Compute $\liminf_{k\to\infty} y_k$.

b) We assume that the sequence of real numbers $(y_k)_{k\in\mathbb{N}}$ converges to a limit $\hat{y}$. Show that $\hat{y} = \liminf_{k\to\infty} y_k$.

c) Let $(y_k)_{k\in\mathbb{N}}$, $(z_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ be two real sequences. Show that

$$\liminf_{k\to\infty} y_k + \liminf_{k\to\infty} z_k \le \liminf_{k\to\infty}(y_k + z_k).$$

In addition, find an example where this inequality is strict.

d) Let I be any index set (possibly infinite, even uncountable), and let $(y_k^i)_{k\in\mathbb{N}} \subset \mathbb{R}$, $i \in I$, be a family of sequences. Show that

$$\sup_{i\in I} \liminf_{k\to\infty} y_k^i \le \liminf_{k\to\infty} \bigl(\sup_{i\in I} y_k^i\bigr).$$

2 *(Lower semi-continuous functions)*

a) Consider the function $f(x) = \sup_{j\in\mathbb{R}} -\exp(-(jx)^2)$. Find an explicit formula for $f(x)$ by evaluating the supremum on the right hand side, and verify that f is lower semi-continuous.

b) More generally, let J be any index set and let $f_j \colon \mathbb{R}^d \to \mathbb{R}$ be lower semi-continuous. Show that the function $f \colon \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ given as

$$f(x) = \sup_{j\in J} f_j(x)$$

is lower semi-continuous.

c) Let $f \colon \mathbb{R}^d \to \mathbb{R}$ be lower semi-continuous and $\alpha \in \mathbb{R}$ arbitrary. Show that the set

$$S_\alpha = \{x \in \mathbb{R}^d : f(x) \le \alpha\}$$

is closed.

3 For the following functions, decide whether they are lower semi-continuous or coercive, and whether they attain a global minimizer:

a) The function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x^4 - 20x^3 + \sup_{k\in\mathbb{N}} \sin(k^2 x).$$

b) The function $g\colon \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = e^x - \frac{1}{x^2+1}.$$

c) The function $h\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$h(x) = x_1^2(1 + x_2^3) + x_1^2.$$

4 Given a matrix $A \in \mathbb{R}^{n\times n}$, we denote by

$$\|A\|_F := \Big(\sum_{i,j=1}^{n} a_{ij}^2\Big)^{1/2}$$

its *Frobenius norm.* Show that the optimisation problem

$$\min_{\substack{A\in\mathbb{R}^{n\times n}\\ \det A>0}} \Big(\|A\|_F + \frac{1}{\det A}\Big)$$

admits a global minimum.

5 *(See N&W, Exercise 2.1).* The *Rosenbrock function* is defined as

$$f(x) := 100(x_2 - x_1^2)^2 + (1 - x_1)^2.$$

a) Compute the gradient and the Hessian of the Rosenbrock function.

b) Show that the point $(1, 1)$ is the unique (global and local) minimizer of f.