NTNU
Norwegian University of Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4180 Optimization I**

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Examination date: 14th May 2019

Examination time (from–to): 09:00–13:00

Permitted examination support material:

- Rottmann, Mathematical formulae.
- Approved basic calculator.

Other information:

- All answers should be justified and include enough details to make it clear which methods or results have been used.
- You may answer to the questions of the exam either in English or in Norwegian.
- Good luck!

Language: English

Number of pages: 2

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
sort/hvit ☒	**farger** ☐
skal ha flervalgskjema ☐	

Date Signature

Problem 1 Consider the unconstrained optimisation problem

$$\min_{x,y} f(x, y),$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$f(x, y) = 2x^2 - 4xy + y^4 + 5y^2 - 10y.$$

a) Determine whether the function f is convex.
(5 points)

b) Find all local and global minimisers of f.
(10 points)

c) Consider now the gradient descent method and the Newton method for the solution of this optimisation problem, with step lengths chosen according to backtracking (Armijo) linesearch. Do these methods converge towards a solution of this optimisation problem? If yes, which of these two methods would you recommend, and why?
(10 points)

d) Perform one step of the gradient descent method with backtracking (Armijo) linesearch starting from the point $(x, y) = (0, 0)$. Start with an initial step length $\alpha = 1$, and use the parameters $c = 0.1$ (sufficient decrease parameter) and $\rho = 0.1$ (contraction factor).
(10 points)

Problem 2 We consider the constrained optimisation problem

$$-x^2 - (y - 2)^2 \to \min$$

subject to the constraint $(x, y) \in \Omega$, where the set Ω is given by the inequalities

$$y \geq 0 \qquad \text{and} \qquad x^2(x + 1) - y \geq 0.$$

a) Sketch the set Ω and determine for each point in Ω whether the LICQ holds.
(5 points)

b) Determine the tangent cone and the cone of linearised feasible directions to the set Ω in the points $(x, y) = (-1, 0)$, $(-2/3, 0)$, and $(0, 0)$.
(10 points)

c) Find all KKT points and all local and global minimisers for this optimisation problem.
(15 points)

Problem 3 For a fixed $t \in \mathbb{R}$, we consider the *elastic net* optimisation problem

$$f(x) = \frac{1}{2}(x-t)^2 + \frac{1}{2}x^2 + |x| \to \min . \tag{P}$$

An equivalent formulation (in the sense that x^* solves (P) if and only if $x^* = y^*$ solve (P')) is the problem

$$\min_{x,y} \frac{1}{2}(x-t)^2 + \frac{1}{2}x^2 + |y| \to \min \qquad \text{subject to } x = y. \tag{P'}$$

a) Formulate the Lagrangian dual of the problem (P') as a constrained optimisation problem.
(10 points)

b) Find an explicit formula for the solution of (P) depending on t.
(10 points)

Problem 4 We consider the optimisation problem

$$f(x) \to \min \qquad \text{s.t. } c(x) \geq 0,$$

where $f\colon \mathbb{R}^d \to \mathbb{R}$ is convex and $\mathcal{C}^1$, and $c\colon \mathbb{R}^d \to \mathbb{R}$ is concave and $\mathcal{C}^1$. Moreover, we assume that the feasible set $\Omega = \left\{x \in \mathbb{R}^d : c(x) \geq 0\right\}$ is non-empty.

a) Formulate the KKT conditions for this optimisation problem and state (with a brief explanation) if they are sufficient and/or necessary optimality conditions.
(5 points)

b) Show that in this situation Slater's constraint qualification is satisfied, if and only if every point $x \in \Omega$ satisfies the LICQ.
(10 points)