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# TMA4180 Optimisation I Spring 2018

## Solutions to exercise set 11

**1** Consider the following linear programming problem:

$$\min c^\top x \qquad \text{subject to } Ax \geq b,\ x \geq 0.$$

Rewrite this program in the standard form. State the dual of the latter, and try to simplify it so that its constraints are stated utilizing matrix $A^\top$.

**Solution:** To write the program in standard form we need to transform the linear inequalities into equalities. To this end let us introduce a vector $\tilde{x} = Ax - b \geq 0$. Then we can restate the problem as follows:

$$\min \begin{bmatrix} c \\ 0 \end{bmatrix}^\top \begin{bmatrix} x \\ \tilde{x} \end{bmatrix},$$
$$\text{subject to } \begin{bmatrix} A & -I \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} = b,\ (x, \tilde{x}) \geq 0.$$

Its dual is

$$\max b^\top y$$
$$\text{subject to } \begin{bmatrix} A^\top \\ -I \end{bmatrix} y + \begin{bmatrix} s \\ \tilde{s} \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix}, (s, \tilde{s}) \geq 0,$$

where we split the variables $(s, \tilde{s})$ in the same fashion as it was done with $(x, \tilde{x})$.

Note that $\tilde{s}$ do not participate in the objective function, and the block-constraints $-Iy + \tilde{s} = 0$, $\tilde{s} \geq 0$ can only be satisfied when $y \geq 0$, since $\tilde{s} = y$ according to the equality constraint. Therefore, the dual simplifies to

$$\max b^\top y$$
$$\text{subject to } A^\top y + s = c, s \geq 0, y \geq 0.$$

In principle, the variables $s$ can also be eliminated from the problem yielding

$$\max b^\top y$$
$$\text{subject to } A^\top y \leq c, y \geq 0.$$

**2** Find the dual of the linear optimisation problem

$$5x_1 + 3x_2 + 4x_3 \to \min \qquad \text{subject to } \begin{cases} x_1 + x_2 + x_3 = 1, \\ \qquad x_i \geq 0, \quad i = 1, 2, 3, \end{cases}$$

and compute its (i.e., the *dual's*) solution.

Using the dual solution, find the primal.

**Solution:** The problem is already stated in the standard form with $c = (5, 3, 4)^\top$, $A = (1, 1, 1)$, and $b = (1)$. Its dual is

$$\begin{aligned} &\max 1y \\ &\text{subject to } \begin{bmatrix}1\\1\\1\end{bmatrix} y + s = \begin{bmatrix}5\\3\\4\end{bmatrix}, s \geq 0. \end{aligned}$$

As in the previous example, the variables $s$ can be eliminated resulting in

$$\begin{aligned} &\max y \\ &\text{subject to } \begin{bmatrix}1\\1\\1\end{bmatrix} y \leq \begin{bmatrix}5\\3\\4\end{bmatrix}. \end{aligned}$$

Clearly this problem is solved by $y^* = 3$, from which we find $s^* = (2, 0, 1)^\top$.

Complementarity condition $x_i^* s_i^* = 0$ implies that $x_1^* = x_3^* = 0$. Then $x_2^*$ can be found from the constraint $Ax = b$, resulting in $x_2^* = 1$.

Note that strong duality holds, that is $c^\top x^* = 3 = b^\top y^*$.

3 Consider the polyhedral set given by the following set of inequalities:

$$\begin{aligned} x_1 &\geq 0, \\ x_2 &\geq 0, \\ x_1 + 2x_2 &\geq 1. \end{aligned}$$

By following the proof of Representation theorem for polyhedral sets (see this note) find the explicit representation of this polyhedral set as a sum of a compact polyhedron (given as a convex combination of its extreme points) and a closed convex cone. A drawing may be a useful tool for further understanding the representation.

**Solution:** The system of constraints can be written as

$$\begin{bmatrix}1 & 0\\0 & 1\\1 & 2\end{bmatrix} x \geq \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Note that the matrix $A$ in the left hand side has rank 2, so that representation theorem applies. The cone $C$ in our case is given by all directions $p$ such that $Ap \geq 0$, resulting in $C = \mathbb{R}^2_+$. The extreme points $v_i$ can be found by considering all $2 \times 2$ non-singular submatrices of $A$, solving the resulting system of equalities, and checking if the resulting point is feasible or not. Let us carry this program.

*Rows 1 and 2* give us the equality system $x_1 = 0$ and $x_2 = 0$, which results in an infeasible (with respect to the last constraint) point.

*Rows 1 and 3* give us the equality system $x_1 = 0$ and $x_1 + 2x_2 = 1$, or $x_1 = 0$, $x_2 = 1/2$. This is a feasible point, and therefore is an extreme point of the polyhedron; let us denote it by $v_1$.

Similarly, *rows 2 and 3* give us the equality system $x_2 = 0$ and $x_1 + 2x_2 = 1$, or $x_1 = 1$, $x_2 = 0$. This is a feasible point, and therefore is an extreme point of the polyhedron; let us denote it by $v_2$.

Thus our polyhedron can be represented as $\{v = \lambda_1 v_1 + \lambda_2 v_2 \mid \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_1 + \lambda_2 = 1\} + \mathbb{R}^2_+$.