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Solutions to exercise set 8

1 For the following two examples, sketch the region Ω defined by the constraints and compute for each point in Ω both the tangent cone and the set of linearized feasible directions. For which points is the LICQ satisfied?

a) The region $\Omega \subset \mathbb{R}^2$ defined by the inequalities

$$y \geq x \qquad \text{and} \qquad y^4 \leq x^3.$$

Solution: We first define constraint functions

$$c_1(x, y) = y - x \quad \text{and} \quad c_2(x, y) = x^3 - y^4,$$

so that $\Omega = \{(x, y) \in \mathbb{R}^2 : c_1(x, y) \geq 0 \text{ and } c_2(x, y) \geq 0\}$, and sketch the region in Figure 1 below.

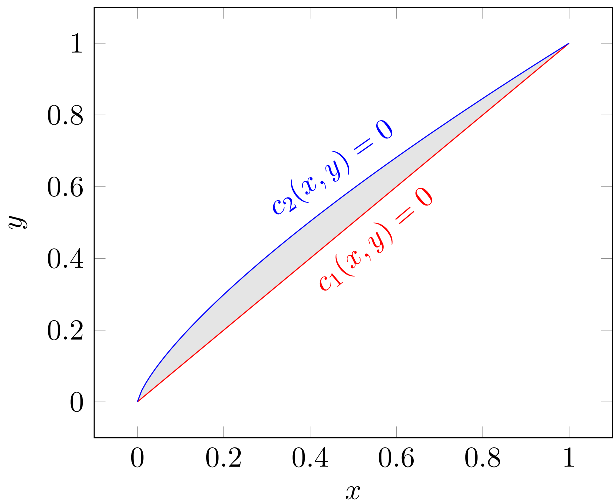

Figure 1: Region Ω in grey, with colors on the boundary specifiying the active constraints.

In order to characterise the tangent cone $T_\Omega(x, y)$ and the set of linearised feasible directions $\mathcal{F}(x, y)$, we employ Lemma 12.2 in N&W, which states that if the LICQ condition holds at a feasible point (x, y), then $T_\Omega(x, y) = \mathcal{F}(x, y)$.

Note first that the LICQ condition holds vacuously in the interior of Ω because all constraints are inactive, and therefore, $T_\Omega(x, y) = \mathcal{F}(x, y) = \mathbb{R}^2$ (why?) at interior points.

Next we consider boundary points with precisely one active constraint. Starting with points for which $c_1(x, y) = 0$—and excluding $(0, 0)$ and $(1, 1)$ where also c_2

is active—we find that $\nabla c_1(x,y) = (-1,1)$. Since $\nabla c_1 \neq 0$, the LICQ condition holds, and so

$$\begin{aligned} T_\Omega(x,y) = \mathcal{F}(x,y) &= \left\{ d \in \mathbb{R}^2 : \nabla c_1(x,y)^\top d \geq 0 \right\} \\ &= \left\{ d \in \mathbb{R}^2 : d_2 \geq d_1 \right\}, \end{aligned}$$

where d is short for (d_1, d_2).

Similarly, if only c_2 is active, we observe that the LICQ condition holds because $\nabla c_2(x,y) = (3x^2, -4y^3) \neq 0$ away from $(0,0)$. This yields

$$\begin{aligned} T_\Omega(x,y) = \mathcal{F}(x,y) &= \left\{ d \in \mathbb{R}^2 : \nabla c_2(x,y)^\top d \geq 0 \right\} \\ &= \left\{ d \in \mathbb{R}^2 : 3x^2 d_1 \geq 4y^3 d_2 \right\}. \end{aligned}$$

Constraint gradients at $(1,1)$ equal $\nabla c_1 = (-1,1)$ and $\nabla c_2 = (3,-4)$, which are linearly independent. Thus the LICQ condition is true, and

$$\begin{aligned} T_\Omega(1,1) = \mathcal{F}(1,1) &= \left\{ d \in \mathbb{R}^2 : \nabla c_1(1,1)^\top d \geq 0 \text{ and } \nabla c_2(1,1)^\top d \geq 0 \right\} \\ &= \left\{ d \in \mathbb{R}^2 : 3d_1 \geq 4d_2 \right\}. \end{aligned}$$

Lastly, since $\nabla c_1(0,0) = (-1,1)$ and $\nabla c_2(0,0) = 0$, the LICQ condition fails at $(0,0)$, and we cannot expect that $T_\Omega(0,0) = \mathcal{F}(0,0)$. Readily,

$$\begin{aligned} \mathcal{F}(0,0) &= \left\{ d \in \mathbb{R}^2 : \nabla c_1(0,0)^\top d \geq 0 \text{ and } \nabla c_2(0,0)^\top d \geq 0 \right\} \\ &= \left\{ d \in \mathbb{R}^2 : d_2 \geq d_1 \right\}. \end{aligned}$$

In order to find the tangent cone, we first consider limiting directions along the constraint boundaries $c_1(x,y) = 0$ and $c_2(x,y) = 0$ as $(x,y) \to (0,0)$. Travelling towards $(0,0)$ when c_1 is active, we may put, using the notation in N&W,

$$z_k = (1/k, 1/k) \quad \text{and} \quad t_k = 1/k,$$

and obtain the limiting direction

$$d = \lim_{k \to \infty} \frac{z_k - (0,0)}{t_k} = (1,1).$$

Note: the length of d is irrelevant; we only care about its direction. Similarly, travelling along $c_2(x,y) = 0$ yields $d = (0,1)$, using for example, the sequences

$$z_k = \left(k^{-1/3}, k^{-1/4}\right) \quad \text{and} \quad t_k = k^{-1/4}.$$

It can furthermore be seen that approaching $(0,0)$ from the interior of Ω gives tangent directions "between" these borderline cases, and so

$$T_\Omega(0,0) = \left\{ d \in \mathbb{R}^2 : d_2 \geq d_1 \geq 0 \right\}.$$

b) The region $\Omega \subset \mathbb{R}^2$ defined by the inequalities

$$y \geq x^4 \qquad \text{and} \qquad y \leq x^3.$$

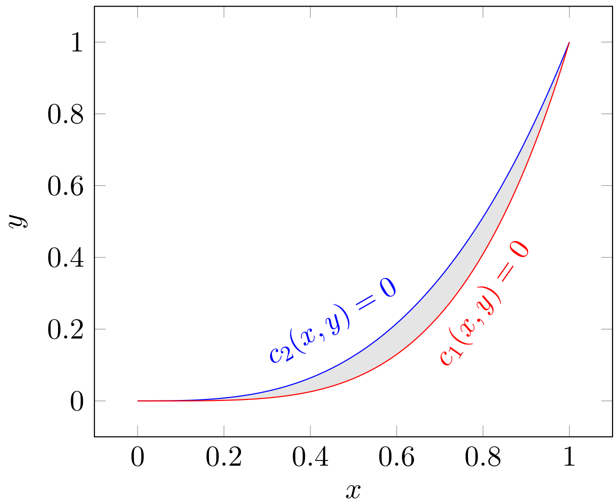

Figure 2: Region Ω in grey, with colors on the boundary specifiying the active constraints.

Solution: Defining

$$c_1(x, y) = y - x^4 \quad \text{and} \quad c_2(x, y) = x^3 - y$$

gives $\Omega = \{(x, y) \in \mathbb{R}^2 : c_1(x, y) \geq 0 \text{ and } c_2(x, y) \geq 0\}$, which is shown in Figure 2.

Omitting details—the process is very similar to the previous question—we obtain that the LICQ condition holds at all feasible points except $(0, 0)$. Moreover, $T_\Omega(x, y) = \mathcal{F}(x, y)$ if (x, y) lies in the interior of Ω;

$$T_\Omega(x, y) = \mathcal{F}(x, y) = \left\{d \in \mathbb{R}^2 : d_2 \geq 4x^3 d_1\right\}$$

when only c_1 is active;

$$T_\Omega(x, y) = \mathcal{F}(x, y) = \left\{d \in \mathbb{R}^2 : 3x^2 d_1 \geq d_2\right\}$$

when only c_2 is active;

$$T_\Omega(1, 1) = \mathcal{F}(1, 1) = \left\{d \in \mathbb{R}^2 : 3d_1 \geq d_2 \geq 4d_1\right\};$$

and

$$\mathcal{F}(0, 0) = \left\{d \in \mathbb{R}^2 : d_2 = 0\right\} \quad \text{and} \quad T_\Omega(0, 0) = \left\{d \in \mathbb{R}^2 : d_2 = 0 \text{ and } d_1 \geq 0\right\}.$$

2 Determine the cone of feasible directions (radial cone) $R_\Omega(0)$, the tangent cone $T_\Omega(0)$ and the cone of linearized feasible directions $F_\Omega(0)$ for the following sets Ω. Determine, which constraint qualifications hold at $0 \in \Omega$.

a) $\Omega = \{\, x \in \mathbb{R}^2 \mid x_1 \geq 0, -(x_1 - 1)^2 - x_2^2 + 1 \geq 0 \,\}$.

Solution: Both constraints are concave, and at $\hat{x} = (1, 0)$ none of the constraints are active. Therefore, Slater's CQ holds and we can expect that $T_\Omega(x) = F_\Omega(x) = \bar{R}_\Omega(x)$ for all $x \in \Omega$.

However, both c_1 and c_2 are active, and $\nabla c_1(0) = (1, 0)^\top$ and $\nabla c_2(0) = (2, 0)^\top$ are linearly dependent. Therefore LICQ does not hold.

Easy computation shows that $F_\Omega(0) = \{ d \in \mathbb{R}^2 \mid d_1 \geq 0 \}$, and $R_\Omega(0) = \{ d \in \mathbb{R}^2 \mid d_1 > 0 \}$. Owing to Slater's CQ we have $T_\Omega(0) = F_\Omega(0) = \bar{R}_\Omega(0) = \{ d \in \mathbb{R}^2 \mid d_1 \geq 0 \}$.

b) $\Omega = \{ x \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0, -x_1 x_2 \geq 0 \}$.

Solution: Slater's CQ does not hold owing to the non-concavity of c_3. All three constraints are active at 0, therefore LICQ cannot not hold in 2D.

It is easy to see that $R_\Omega(0) = T_\Omega(0) = \Omega$ (which is not a convex cone). Therefore, necessarily, $F_\Omega(0) \neq T_\Omega(0)$. $\nabla c_1(0) = (1, 0)^\top$, $\nabla c_2(0) = (0, 1)^\top$, $\nabla c_3(0) = (0, 0)^\top$, and therefore $F_\Omega(0) = \{ d \in \mathbb{R}^2 \mid d_1 \geq 0, d_2 \geq 0 \} = \mathbb{R}^2_+$.

c) $\Omega = \{ x \in \mathbb{R}^2 \mid x_1^3 - x_2 \geq 0, x_2 - x_1^5 \geq 0, x_2 \geq 0 \}$.

Solution: Slater's CQ cannot hold because the constraints are not concave. At $x = 0$ all three constraints are active, therefore LICQ cannot hold (in two dimensions).

$\nabla c_1(0) = (0, -1)^\top$; $\nabla c_2(0) = (0, 1)^\top$; $\nabla c_3(0) = (0, 1)^\top$, and therefore $F_\Omega(0) = \{ d \in \mathbb{R}^2 \mid d_2 = 0 \}$.

$R_\Omega(0) = \{ 0 \}$, owing to the presence of the non-linear constraints c_1 and c_2.

Finally, the tangent cone is given by $F_\Omega(0) = \{ d \in \mathbb{R}^2 \mid d_1 \geq 0, d_2 = 0 \}$.

d) $\Omega = \{ x \in \mathbb{R}^2 \mid x_2 \geq 0, (x_1 - 1)^2 + x_2^2 - 1 = 0 \}$.

Solution: Contrary to the situation in a), the second constraint is now a non-linear equality constraint, and therefore Slater's CQ cannot hold.

LICQ does not hold either, exactly as in a).

Finally, per definition $F_\Omega(0) = \{ d \in \mathbb{R}^2 \mid \nabla c_1(0)^\top d \geq 0, \nabla c_1(0)^\top d = 0 \} = \{ d \in \mathbb{R}^2 \mid d_1 = 0 \}$, $R_\Omega(0) = \{ 0 \}$ as in c), owing to the presence of the non-liner equality constraint. Direct computation shows that $T_\Omega(0) = \{ d \in \mathbb{R}^2 \mid d_1 = 0 \}$.

Now consider the function $f(x) = x_1$.

e) Find the points of global minimum for f over Ω given in examples a)–d). Verify that the optimality conditions $\forall p \in T_\Omega(0) : \nabla f(0)^\top p \geq 0$ are satisfied.

Solution: Graphically one can see that in all examples except b) the point 0 is the only point of global minimum, whereas in b) any point $x \in \{0\} \times \mathbb{R}_+$ is a point of global minimum.

Since $\nabla f(0) = (1, 0)^\top$, and in all cases $p \in T_\Omega(0) \implies p_1 \geq 0$, the optimality condition holds.