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TMA4180 Optimisation I Spring 2018

Solutions to exercise set 1

1 **a)** Consider the sequence $x_k = (k \mod 3) - 1/k$, where $k \mod 3 \in \{0, 1, 2\}$ is the remainder of integer division of k by 3. Does this sequence converge? Compute $\liminf_{k\to\infty} x_k$.

Solution: The sequence cannot converge because it is not Cauchy: indeed the difference between the consecutive elements of the sequence is at least 1 because of the term $k \mod 3$.
However, we can directly compute

$$\begin{aligned}\liminf_{k\to\infty} x_k &= \lim_{k\to\infty} \inf_{n\geq k} [(n \mod 3) - 1/n] \\ &= \lim_{k\to\infty} \inf_{n\in\{k,k+1,k+2\}} [(n \mod 3) - 1/n] = 0,\end{aligned} \tag{1}$$

because the last infimum equals $-1/k$, $-1/(k+1)$, or $-1/(k+2)$ depending on whether the remainder $k \mod 3$ is 0, 2, or 1. In any case, the limit as $k \to \infty$ is 0.

b) Suppose that the sequence of real numbers $(x_k)_{k=1}^{\infty}$ converges the limit $\bar{x} \in \mathbb{R}$. Show that $\liminf_{k\to\infty} x_k = \bar{x}$.

Solution: For any $\varepsilon > 0$ there is $N \in \mathbb{N}$, such that $\forall n \geq N$ we have the inequalities $\bar{x} - \varepsilon < x_n < \bar{x} + \varepsilon$. As a consequence, $\bar{x} - \varepsilon \leq \inf_{n\geq N} x_n \leq \bar{x} + \varepsilon$, and subsequently $\lim_{k\to\infty} \inf_{n\geq k} x_n = \bar{x}$.

c) Let $(y_k)_{k\in\mathbb{N}}$, $(z_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ be two real sequences. Show that

$$\liminf_{k\to\infty} y_k + \liminf_{k\to\infty} z_k \leq \liminf_{k\to\infty} (y_k + z_k).$$

In addition, find an example where this inequality is strict.

Solution: For every $k \in \mathbb{N}$ we have the obvious inequalities $x_k \geq \inf_{n\geq k} x_n$ and $y_k \geq \inf_{n\geq k} y_n$. Note that the right hand side of these inequalities is a non-decreasing function of k, which allows us to add these inequalities, take an infimum over $k \geq m$, and actually compute this infimum in the right hand side:

$$\inf_{k\geq m} [x_k + y_k] \geq \inf_{k\geq m} [\inf_{n\geq k} x_n + \inf_{n\geq k} y_n] = \inf_{n\geq m} x_n + \inf_{n\geq m} y_n.$$

It only remains to take $\lim_{m\to\infty}$ on both sides of this inequality; the limits exist because all sequences are non-decreasing.
Strict inequality occurs for example if $y_k = (-1)^k$ and $z_k = (-1)^{k+1}$. Then $y_k + z_k = 0$ for all k, which yields

$$\liminf_{k\to\infty} y_k = -1 = \liminf_{k\to\infty} z_k \qquad \text{and} \qquad \liminf_{k\to\infty} (y_k + z_k) = 0.$$

d) Let I be any index set (possibly infinite, possibly uncountable), and let $(y_k^i)_{k\in\mathbb{N}} \subset \mathbb{R}$, $i \in I$, be a family of sequences. Show that

$$\sup_{i\in I} \liminf_{k\to\infty} y_k^i \le \liminf_{k\to\infty} \big(\sup_{i\in I} y_k^i\big).$$

Solution: For each $k \in \mathbb{N}$ and every $i \in I$ we have the inequality $y_k^i \le \sup_{j\in I} y_k^j$. Taking first the infimum, and then the limit on both sides of this inequality we conclude that for every $i \in I$,

$$\liminf_{k\to\infty} y_k^i \le \liminf_{k\to\infty} \sup_{j\in I} y_k^j.$$

It only remains to take a supremum over $i \in I$ on the both sides of this inequality, while noticing that the right hand side is independent of i.

2 **a)** Consider the function $f(x) = \sup_{i\in\mathbb{R}} - \exp(-(ix)^2)$. Determine the explicit formula for $f(x)$ by evaluating the supremum on the right hand side, and check that f is lower semi-continuous.

Solution: Direct computation shows that

$$f(x) = \begin{cases} -1, & \text{if } x = 0, \\ 0, & \text{otherwise.} \end{cases}$$

It is not difficult to see that this function is lower semi-continuous by checking the definition.

b) More generally, let I be an index set and let $f_i \colon \mathbb{R}^n \to \mathbb{R}$ be lower semi-continuous. Show that the function $f \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f(x) = \sup_{i\in I} f_i(x)$$

is lower semi-continuous.

Solution: This can be viewed as a direct consequence of exercise 1 d). Indeed, let $\bar{x} \in \mathbb{R}^n$ be an arbitrary point, and let further consider an arbitrary sequence $(x_k)_{k=1}^{\infty} \in \mathbb{R}^n$ converging to $\bar{x}$. Since each f_i is l.s.c., it follows that $f_i(x) \le \liminf_{k\to\infty} f_i(x_k)$, and as a result also

$$\sup_{i\in I} f_i(x) \le \sup_{i\in I} \Big(\liminf_{k\to\infty} f_i(x_k) \Big) \le \liminf_{k\to\infty} \sup_{i\in I} f_i(x_k),$$

where the last inequality is owing to exercise 1 d). It remains to identify the left hand side of the inequality above with $f(x)$ and the right hand side with $\liminf_{k\to\infty} f(x_k)$.

c) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a lower semi-continuous function, and $\alpha \in \mathbb{R}$ be an arbitrary number. Show that the set $S_\alpha = \{\, x \in \mathbb{R}^n \mid f(x) \le \alpha \,\}$ is closed.

Solution: Consider a sequence of points $(x_k)_{k=1}^{\infty}$, such that each $x_k \in S_\alpha$ and $\lim_{k\to\infty} x_k = \bar{x}$, for some $\bar{x} \in \mathbb{R}^n$. We need to show that $\bar{x} \in S_\alpha$.

This follows from the following string of inequalities:

$$f(\bar{x}) \leq \liminf_{k\to\infty} f(x_k) \leq \alpha,$$

where the first inequality is owing to the lower semicontinuity of f, and the second is owing to the inequality $f(x_k) \leq \alpha$.

3 For the following functions, decide whether they are lower semi-continuous and/or coercive, and whether they attain a global minimizer:

a) The function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x^4 - 20x^3 + \sup_{k\in\mathbb{N}} \sin(kx).$$

Solution: f is a sume of the continuous (hence l.s.c.) funciton $x^4 - 20x^3$ and a l.s.c. function $\sup_{k\in\mathbb{N}} \sin(kx)$ (exercise 2 b). The sum is l.s.c. owing to exercise 1 c) (the proof is similar to that in exercise 2 b).
The function is coercice, because x^4 is, and it dominates all other terms for large $|x|$.
As a result, the global minimum is attained (see Theorem 9 in the note "Minimisers of Optimization Problems.")

b) The function $g\colon \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = e^x - \frac{1}{x^2+1}.$$

Solution: Again, the function is continuous, hence also l.s.c.
It is not coercive, because $lim_{x\to-\infty} g(x) = 0 \neq +\infty$.
On the other hand, the closed set $S = \{\, x \in \mathbb{R} \mid g(x) \leq g(-1) \,\}$ is bounded (and therefore compact), since $g(-1) < 0 = \lim_{x\to-\infty} g(x)$ and $g(-1) < 0 < +\infty = \lim_{x\to+\infty} g(x)$.
Therefore, the global minimum of g is attained over S, and this is also a global minimum of g over $\mathbb{R}$.

c) The function $h\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$h(x) = x_1^2(1 + x_2^3) + x_1^2.$$

Solution: The function is continuous (polynomial, in fact) - hence also l.s.c.
It is not coercive, since for any fixed $x_1 \neq 0$ and $x_2 \to -\infty$ we have $h(x) \to -\infty$.
This also shows that the function is unbounded from below, and as a result the global minimum is not attained.

4 Given a matrix $A \in \mathbb{R}^{n\times n}$, we denote by

$$\|A\|_F := \Big(\sum_{i,j=1}^{n} a_{ij}^2\Big)^{1/2}$$

its *Frobenius norm*. Show that the optimisation problem

$$\min_{\substack{A\in\mathbb{R}^{n\times n}\\ \det A>0}} \Big(\|A\|_F + \frac{1}{\det A}\Big)$$

admits a global minimum.

Solution: In order to guarantee existence of solutions, we need to arrive at a situation where we minimize a l.s.c. function over a compact set.

The Frobenius norm is continuous, and the determinant is a continuous (in fact, polynomial) function of matrix elements. Thus the objective function $f(A) = \|A\|_{\mathrm{F}} + (\det A)^{-1}$ is continuous over the set $\Omega = \{A \in \mathbb{R}^{n\times n} : \det A > 0\}$.

However, the feasible set Ω is neither bounded nor closed!

Thus we should try to construct a closed and bounded (compact) set inside Ω, on which the global minimum of f over Ω is attained.

The identity matrix $I \in \Omega$, with $f(I) = n^{1/2} + 1$. Therefore, the global minimum, if attained, must lie inside $\{\, A \in \Omega \mid f(A) \le f(I) = n^{1/2} + 1 \,\} \subset \{\, A \in \mathbb{R}^{n\times n} \mid \|A\|_{\mathrm{F}} \le n^{1/2} + 1 \,\} \cap \{\, A \in \Omega \mid \det(A) \ge (n^{1/2} + 1)^{-1} \,\} =: S$.

The latter set is bounded (the first set in the intersection is bounded) and closed (both intersected sets are closed, since the functions defining the inequality constraints are continuous).

Thus Proposition 7 from "Minimisers of Optimization Problems" is applicable and f attauns a global min over S, hence also over Ω.

5 *(See N&W, Exercise 2.1)*. The *Rosenbrock function* is defined as

$$f(x) := 100(x_2 - x_1^2)^2 + (1 - x_1)^2.$$

a) Compute the gradient and the Hessian of the Rosenbrock function.

Solution: Routine differentiation yields

$$\nabla f(x) = \begin{bmatrix} -400x_1(x_2 - x_1^2) - 2(1 - x_1) \\ 200(x_2 - x_1^2) \end{bmatrix}$$

and

$$\nabla^2 f(x) = \begin{bmatrix} 1200x_1^2 - 400x_2 + 2 & -400x_1 \\ -400x_1 & 200 \end{bmatrix}.$$

b) Show that the point $(1, 1)$ is the unique (global and local) minimizer of f.

Solution: Since f is defined in terms of quadratic terms, it is bounded from below by 0. Moreover $f(x) = 0$ if and only if $x_2 = x_1^2$ and $1 - x_1 = 0$, which means that $x_1 = 1$ and $x_2 = 1$. Therefore $(1, 1)$ is the unique minimiser of f.

Another argument: $(1, 1)$ is the only stationary/critial point of f, that is, the only point for which $\nabla f = 0$. Moreover, the Hessian

$$\nabla^2 f(1, 1) = \begin{bmatrix} 802 & -400 \\ -400 & 200 \end{bmatrix}$$

has eigenvalues $501 \pm \sqrt{250601} > 0$, and so f is symmetric positive definite at $(1, 1)$. Hence, $(1, 1)$ is a strict local minimiser by Theorem 2.4 in N&W. As f is coercive (check it), $(1, 1)$ must be a global minimiser, and in fact, *the* minimiser since there are no other stationary points.