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Exercise set 9

1 Consider a constrained minimization problem with continuously differentiable objective function and constraints. Which of the following statements are true?

a) x^* is a (point of) global minimum $\implies$ x^* is a KKT point.

b) x^* is a local minimum and CQ holds $\implies$ x^* is a KKT point.

c) x^* is a KKT point and CQ holds $\implies$ x^* is a local minimum.

d) x^* is a global minimum and the problem is convex $\implies$ x^* is a KKT point.

e) x^* is a KKT point and the problem is convex $\implies$ x^* is a global minimum.

2 Consider the constrained optimization problem

$$x^2 + y^2 \to \min \qquad \text{such that} \qquad \begin{cases} x + y \geq 1, \\ y \leq 2, \\ y^2 \geq x. \end{cases}$$

a) Formulate the KKT-conditions for this optimization problem.

b) Find all KKT points for this optimization problem.

c) Find all local and global minima for this optimization problem.

(Part b can be very tedious. One strategy is to consider all possible active sets and determine for each active set whether KKT-points exist. It can also be extremely helpful to sketch the feasible set and the function.)

3 Consider a *linear programming problem*

$$c^\top x \to \min \qquad \text{subject to} \qquad Ax \geq b,$$

where $c, x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and $A \in \mathbb{R}^{m\times n}$.

a) Show that for each feasible point x the radial cone $R_\Omega(x)$ is the same as the cone of linearized feasible directions $F_\Omega(x)$. Conclude that $T_\Omega(x) = F_\Omega(x)$, i.e., CQ holds.

b) State the KKT optimality conditions for this problem. Show that at every KKT point x we have the equality $c^\top x = b^\top \lambda$, where $\lambda \in \mathbb{R}^m_+$ is a vector of Lagrange multipliers.

4 Consider the constrained optimization problem

$$x \to \min \qquad \text{such that} \qquad \begin{cases} y \geq x^4, \\ y \leq x^3. \end{cases}$$

Find all KKT points and local minima for this optimization problem.

5 Consider the constrained optimization problem

$$xy \to \min \qquad \text{such that} \qquad \begin{cases} y \geq x, \\ y^4 \leq x^3. \end{cases}$$

a) Find all KKT points and local minima for this optimization problem.

b) Compute the critical cone at $(0, 0)$, and show that there exist directions d contained in the critical cone for which $d^\top \nabla^2 \mathcal{L}((0, 0); \lambda^*)d < 0$.

c) Show that $d^\top \nabla^2 \mathcal{L}((0, 0); \lambda^*)d \geq 0$ for all vectors d contained in the tangent cone to the feasible set at $(0, 0)$.