Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4180 Optimisation I Spring 2018

Solutions to exercise set 7

1 We will consider sets $\Omega_1 = \{ x \in \mathbb{R}^n \mid |x_i| \leq 1 \} = \{x \in \mathbb{R}^n \mid \|x\|_\infty \leq 1 \}$, $\Omega_2 = \{ x \in \mathbb{R}^n \mid \|x\|_2 \leq 1 \}$, and $\Omega_3 = \{ x \in \mathbb{R}^n \mid \|x\|_2 = 1 \}$.

a) Show that the sets Ω_1 and Ω_2 are non-empty, convex and closed.

b) In 2D, determine the normal cones $N_{\Omega_i}(x)$ and radial cones (cones of feasible directions) $R_{\Omega_i}(x)$ at $x^1 = (0, 1)$ for $i = 1, 2$ In case of Ω_2 do the same at $x^2 = (1, 1)$.
It may be easier to start with a sketch.

c) Show that the radial cone $R_{\Omega_3}(x) = \{0\}$ for all $x \in \Omega_3$. Owing to the symmetry it is sufficient to analyse only one such point.

d) Consider the projection problem

$$\pi_{\Omega_i}(z) = \arg\min_{y \in \Omega_i} f(y) := \frac{1}{2}\|y - z\|_2^2.$$

Use the necessary and sufficient optimality conditions $\nabla f(\pi_{\Omega_i}(z))^\top (x - \pi_{\Omega_i}(z)) \geq 0$, $\forall x \in \Omega$, to verify the projection formulae $\pi_{\Omega_1}(z) = \min\{1, \max\{-1, z\}\}$, where min and max is applied component-wise to the vector, and $\pi_{\Omega_2}(z) = z / \max\{1, \|z\|_2\}$.

e) Consider a two-dimensional situation, and put $f(x) = x_1^2 + (x_2 + 2)^2$. Let $x^{(0)} = (1, -1)$. Find the point of global minimum for this function over Ω_1. (you can do this graphically). Compute one step of the projected gradient method $x^{(1)} = \pi_{\Omega_1}[x^{(0)} - \alpha \nabla f(x^{(0)})]$ using steplength $\alpha = 1$.

2 Consider a convex set $\Omega \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$. Show that the radial cone (cone of feasible directions) $R_\Omega(x)$ is convex.

3 **a)** Consider a set Ω described using linear *equality* constraints, $\Omega = \{ z \in \mathbb{R}^n \mid a_i^\top z = b_i \}$, $i = 1, \dots, m$. Express the radial cone $R_\Omega(x)$ and the normal cone $N_\Omega(x)$ using the vectors $a_i \in \mathbb{R}^n$, $i = 1, \dots, m$ for $x \in \Omega$.

b) Consider a set Ω described using linear *inequality* constraints, $\Omega = \{ z \in \mathbb{R}^n \mid a_i^\top z \leq b_i \}$, $i = 1, \dots, m$. Given $x \in \Omega$ let $\mathcal{A}(x) \subset \{1, \dots, m\}$ be the set of active, or binding constraints: $a_i^\top x = b_i$, $i \in \mathcal{A}(x)$, $a_i^\top x < b_i$, $i \notin \mathcal{A}(x)$. Express the radial cone $R_\Omega(x)$ using the vectors $a_i \in \mathbb{R}^n$, $i = 1, \dots, m$, and $\mathcal{A}(x)$. Show that $N_\Omega(x) \supseteq \{ \sum_{i \in \mathcal{A}} \lambda_i a_i \mid \lambda_i \geq 0 \}$. (In fact the last inclusion is equality, but it is a bit more difficult to prove.)