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TMA4180
Optimization I
Spring 2018

Exercise set 3

1 Consider the quadratic function

$$f(x) = \tfrac{1}{2}x^{\mathrm{T}}Ax - b^{\mathrm{T}}x,$$

where $A \in \mathbb{R}^{n\times n}$ is a symmetric and positive definite matrix and $b \in \mathbb{R}^n$.

a) Let $p \in \mathbb{R}^n$ be a direction satisfying the inequality $\nabla f(x)^{\mathrm{T}}p < 0$. Compute analytically the steplength $\alpha_{x,p}$, which solves the linesearch problem $\min_{\alpha>0} f(x + \alpha p)$

b) Let $x, p \in \mathbb{R}^n$ and $\alpha_{x,p} > 0$ be as in the previous question. Show that the steplength $\alpha_{x,p}$ satisfies the strong Wolfe conditions if and only if $c_1 \leq 1/2$.

c) Let $A = Q\Lambda Q^{\mathrm{T}}$ be the eigenvalue decomposition of A, where Λ is a diagonal matrix with eigenvalues on the diagonal, and columns of Q are the orthonormal eigenvectors of A. In particular, $Q^{\mathrm{T}}Q = I$, where $I \in \mathbb{R}^{n\times n}$ is the identity matrix.

Show that applying the steepest descent method with exact linesearch to the problem $\min_{x\in\mathbb{R}^n} 0.5x^{\mathrm{T}}Ax - b^{\mathrm{T}}x$ is equivalent to applying the steepest descent method with exact linesearch to $\min_{y\in\mathbb{R}^n} 0.5y^{\mathrm{T}}\Lambda y$, in the following sense: if $x_0 = Qy_0 + A^{-1}b$ then the iterates generated by the two methods satisfy the same relation, $x_k = Qy_k + A^{-1}b$, $k \geq 1$.

In this sense, the behaviour of the steepest descent method is insensitive with respect to translation or orthogonal transformation of coordinates.

2 Let f be twice continuously differentiable in a vicinity of $x_0 \in \mathbb{R}^n$. Assume that $\nabla^2 f(x_0)$ is positive definite and consider the Newton's direction $p_x = -[\nabla^2 f(x_0)]^{-1}\nabla f(x_0)$ together with the unit Newton's step $x_1 = x_0 + p_x$.

Let us now perform an affine transformation (translation, rotation, and scaling) of coordinates $x = By + c$, where $B \in \mathbb{R}^{n\times n}$ is a non-singular matrix (not necessarily orthogonal), and $c \in \mathbb{R}^n$ is some vector. Demonstrate that Newton's method is insensitive with respect to such transformations: that is, if $g(y) = f(By + c) = f(x)$, $x_0 = By_0 + c$, and finally $y_1 = y_0 - [\nabla^2 g(y_0)]^{-1}\nabla g(y_0)$ then $x_1 = By_1 + c$.

3 Let $A \in \mathbb{R}^{n\times n}$ be an SPD matrix with the eigenvalue decomposition $A = Q\Lambda Q^T$, and let $b \in \mathbb{R}^n$ be an arbitrary vector. We put $x^* = A^{-1}b$ to be the optimal solution of the quadratic unconstrained minimization problem $\min_{x\in\mathbb{R}^n} 0.5x^{\mathrm{T}}Ax - b^{\mathrm{T}}x$. Suppose that the eigenvaluse of A are sorted as $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. During the lecture we have discussed that for starting point of the type $x_0 = x^* + \lambda_1^{-1}q_1 + \lambda_n^{-1}q_n$, where

q_i are orthonormal eigenvectors of A (columns of Q) corresponding to eigenvalues λ_i, the steepest descent method with exact linesearch for this problem generates iterates satisfying

$$\|x_k - x^*\| = \left(\frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1}\right)^k \|x_0 - x^*\|,$$

which converges to zero linearly, and arbitrarily slowly for large condition numbers $\mathrm{cond}(A) = \lambda_n/\lambda_1$. Approximately, the number of iterations needed to achieve some prescribed tolerance scales proportionally to the condition number of A.

a) Implement the steepest descent method with exact linesearch for this problem and verify the estimate above numerically.

Hint: one can generate random positive definite matrices for example as follows:

```
import numpy as np
N = 10
# generate NxN random matrix
X = np.random.randn(N,N)
# generate NxN orthogonal matrix from it
Q = np.linalg.qr(X)[0]
# generate some random eigenvalues between lam_min and lam_max
lam_min   = 1.0
lam_max   = 100.0
lmbda     = lam_min + (lam_max-lam_min)*np.sort(np.random.rand(N))
lmbda[0]  = lam_min
lmbda[-1]=lam_max
Lambda   = np.diag(lmbda)
A = np.matmul(Q,np.matmul(Lambda,Q.T))
# random vector
b = np.random.randn(N)
# A^{-1}b
xstar  = np.linalg.solve(A,b)
# starting point
x0     = xstar + 1.0/lmbda[0]*Q[:,0] + 1.0/lmbda[-1]*Q[:,-1]
```

b) Not everyone has given up on the steepest descent method. Consider for example the following accelerated version of the method due to Nesterov:

$$\begin{aligned} p_k &= -\nabla f(x_k), \\ y_{k+1} &= x_k + \lambda_n^{-1} p_k, \\ x_{k+1} &= s_1 y_{k+1} + s_0 y_k, \end{aligned}$$

where we put $y_0 = x_0$, $s_0 = -(\lambda_n^{1/2} - \lambda_1^{1/2})/(\lambda_n^{1/2} + \lambda_1^{1/2})$, and $s_1 = 1.0 - s_0$.

Implement this method and verify numerically, that the number of iterations needed to achieve some prescribed tolerance scales proportionally to the square root of the condition number of A, $\lambda_n^{1/2}/\lambda_1^{1/2}$.

4 Implement both the steepest descent method and the Newton's method with linesearch satisfying Wolfe conditions (use a bisection algorithm for this).

Apply the method to minimizing the Rosenbrock function:

$$f(x,y) := 100(y - x^2)^2 + (1 - x)^2.$$

As Newtons direction is not necessarily a descent direction, we can simply use the steepest descent direction when the following inequality holds:

$$-\nabla f(x_k)^{\mathrm{T}} p_k^{\text{Newton}} \leq \varepsilon \|\nabla f(x_k)\| \|p_k^{\text{Newton}}\|,$$

that is, when the angle between the Newton's direction and the steepest descent direction gets dangerously close to $\pi/2$ or exceeds this value.

Verify numerically that the unit Newton's steps are accepted by the linesearch algorithm provided that the sufficient decrease parameter satisfies the inequality $0 < c_1 < 1/2$.