Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4180
Optimisation I
Spring 2018

Solutions to exercise set 10

1 Consider the constrained optimisation problem

$$\frac{1}{2}(x^2+y^2) \to \min \qquad \text{subject to } xy = 1.$$

a) Find—by whatever means—the solutions of this problem. In addition, find the values of the corresponding Lagrange multipliers.

b) Formulate the unconstrained optimisation problem that results from the application of the quadratic penalty method with parameter $\mu > 0$. Solve these problems for all possible parameters μ and verify that the solutions converge to the solutions of the constrained optimization problem as $\mu \to \infty$.

c) Formulate the augmented Lagrangian for this constrained optimization problem and find (for all possible parameters $\lambda \in \mathbb{R}$ and $\mu > 0$) the global solutions of this (unconstrained) optimization problem. For which parameters does one recover the solution of the original constrained problem?

d) The ℓ^1-penalty function for this optimisation problem is defined, for some parameter $\mu > 1$, as

$$\Phi_1(x,y;\mu) := \frac{1}{2}(x^2+y^2) + \mu|xy-1|.$$

Find for each parameter $\mu > 0$ the global minimisers of this function. For which parameters $\mu > 0$ do they coincide with the solutions of the original problem?

2 Consider the constrained optimisation problem

$$x + y \to \min \qquad \text{subject to } x^2+y^2 \le 1.$$

Formulate a logarithmic barrier method for the solution of this constrained optimisation problem and compute its solution for each parameter $\mu > 0$ in the barrier functional.

3 Consider the quadratic programming problem

$$\min (x+1)^2 + (y+1)^2, \qquad \text{subject to} \quad \begin{aligned} x+y &\ge 0, \\ -x &\ge -2, \\ -y &\ge -2. \end{aligned}$$

a) Find the global minimum for this optimization problem (you can do this graphically). Determine the corresponding active set and Lagrange multipliers.

b) Solve the problem using an active set method. Start with $(x_0, y_0) = (2, 0)$ and working set $W_0 = \{2\}$.

4 Assume that $A \in \mathbb{R}^{m \times n}$ with $m < n$ is a matrix of full rank and that $b \in \mathbb{R}^m \setminus \{0\}$. Consider the optimisation problem

$$\frac{1}{2}\|x\|^2 \to \min \qquad \text{subject to} \qquad Ax = b. \tag{1}$$

(See also problem 3 in exercise set 7.)

a) Formulate the augmented Lagrangian $\mathcal{L}_A$ for problem (1), and find for all possible parameters $\lambda \in \mathbb{R}^m$ and $\mu > 0$ a formula for the global solutions of the resulting unconstrained optimisation problem.

b) For which parameters $\lambda \in \mathbb{R}^m$ and $\mu > 0$ is the minimiser of the augmented Lagrangian equal to the solution of (1)?

c) An iterative algorithm for the solution of (1) using the augmented Lagrangian may have the form

$$\begin{aligned} x^{k+1} &\in \arg\min_x \mathcal{L}_A(x, \lambda^k; \mu), \\ \lambda^{k+1} &= \lambda^k - \mu(Ax^{k+1} - b). \end{aligned}$$

Show that this iteration converges for all initial values $x^0 \in \mathbb{R}^n$, $\lambda^0 \in \mathbb{R}^m$, and all $\mu > 0$ to the unique solution of (1).[1]

Hint: Interpret the iteration for the Lagrange parameter λ as a fixed-point iteration—by using the explicit formula for x^{k+1} derived in the first part of this exercise—and then use results from previous numerics courses to show that this fixed-point scheme converges.

Useful matrix formula: if B, C, and $B + C$ are invertible matrices of the same size, then

$$\left(B^{-1} + C^{-1}\right)^{-1} = B(B + C)^{-1}C = C(B + C)^{-1}B.$$

[1] This problem does not completely fall within the curriculum of this optimisation class, but it is still recommended to *try* to solve it.