



1 (*Properties of the lower limit*)

- a) Consider the sequence $x_k = (k \bmod 3) - 1/k$, where $k \bmod 3 \in \{0, 1, 2\}$ is the remainder of integer division of k by 3. Does this sequence converge? Compute $\liminf_{k \rightarrow \infty} x_k$.
- b) Suppose that the sequence of real numbers $(x_k)_{k=1}^{\infty}$ converges the limit $\bar{x} \in \mathbb{R}$. Show that $\liminf_{k \rightarrow \infty} x_k = \bar{x}$.
- c) Let $(y_k)_{k \in \mathbb{N}}, (z_k)_{k \in \mathbb{N}} \subset \mathbb{R}$ be two real sequences. Show that

$$\liminf_{k \rightarrow \infty} y_k + \liminf_{k \rightarrow \infty} z_k \leq \liminf_{k \rightarrow \infty} (y_k + z_k).$$

In addition, find an example where this inequality is strict.

- d) Let I be any index set (possibly infinite, possibly uncountable), and let $(y_k^i)_{k \in \mathbb{N}} \subset \mathbb{R}$, $i \in I$, be a family of sequences. Show that

$$\sup_{i \in I} \liminf_{k \rightarrow \infty} y_k^i \leq \liminf_{k \rightarrow \infty} (\sup_{i \in I} y_k^i).$$

2 (*Lower semi-continuous functions*)

- a) Consider the function $f(x) = \sup_{i \in \mathbb{R}} -\exp(-(ix)^2)$. Determine the explicit formula for $f(x)$ by evaluating the supremum on the right hand side, and check that f is lower semi-continuous.
- b) More generally, let I be an index set and let $f_i: \mathbb{R}^n \rightarrow \mathbb{R}$ be lower semi-continuous. Show that the function $f: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$f(x) = \sup_{i \in I} f_i(x)$$

is lower semi-continuous.

- c) Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a lower semi-continuous function, and $\alpha \in \mathbb{R}$ be an arbitrary number. Show that the set $S_\alpha = \{x \in \mathbb{R}^n \mid f(x) \leq \alpha\}$ is closed.

3 For the following functions, decide whether they are lower semi-continuous and/or coercive, and whether they attain a global minimizer:

- a) The function $f: \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(x) = x^4 - 20x^3 + \sup_{k \in \mathbb{N}} \sin(kx).$$

b) The function $g: \mathbb{R} \rightarrow \mathbb{R}$ given by

$$g(x) = e^x - \frac{1}{x^2 + 1}.$$

c) The function $h: \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$h(x) = x_1^2(1 + x_2^3) + x_1^2.$$

4 Given a matrix $A \in \mathbb{R}^{n \times n}$, we denote by

$$\|A\|_F := \left(\sum_{i,j=1}^n a_{ij}^2 \right)^{1/2}$$

its *Frobenius norm*. Show that the optimisation problem

$$\min_{\substack{A \in \mathbb{R}^{n \times n} \\ \det A > 0}} \left(\|A\|_F + \frac{1}{\det A} \right)$$

admits a global minimum.

5 (See *N&W, Exercise 2.1*). The *Rosenbrock function* is defined as

$$f(x) := 100(x_2 - x_1^2)^2 + (1 - x_1)^2.$$

a) Compute the gradient and the Hessian of the Rosenbrock function.

b) Show that the point $(1, 1)$ is the unique (global and local) minimizer of f .