Norwegian University of Science and Technology
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TMA4180 Optimisation I Spring 2017

Solutions to exercise set 9

1 **a)** We start by minimising the Lagrangian

$$\mathcal{L}(x,\lambda) = \tfrac{1}{2}\|x\|^2 - \lambda^\top(Ax-b),$$

where $\lambda \in \mathbb{R}^m$, with respect to x. Calculating

$$\nabla_x \mathcal{L}(x,\lambda) = x - A^\top\lambda \qquad \text{and} \qquad \nabla_x^2 \mathcal{L}(x,\lambda) = \mathrm{Id}_{n\times n}$$

shows that $\mathcal{L}(\cdot,\lambda)$ is positive definite and has a unique minimiser $x^* = A^\top\lambda$. Thus the dual problem is

$$\max_{\lambda\in\mathbb{R}^m} q(\lambda),$$

where

$$\begin{aligned} q(\lambda) = \min_{x\in\mathbb{R}^n} \mathcal{L}(x,\lambda) &= \mathcal{L}(x^*,\lambda) \\ &= \tfrac{1}{2}\|A^\top\lambda\|^2 - \lambda^\top(AA^\top\lambda - b) \\ &= b^\top\lambda - \tfrac{1}{2}\|A^\top\lambda\|^2. \end{aligned}$$

b) Observe first that

$$\nabla q(\lambda) = b - (A^\top)^\top A^\top\lambda = b - AA^\top\lambda$$

and

$$\nabla^2 q(\lambda) = -AA^\top.$$

Since A has full rank, $AA^\top$ is positive definite. Hence, q is negative definite, and there exists a unique maximiser λ^* of the dual problem satisfying $\nabla q(\lambda^*) = 0$, that is, $AA^\top\lambda^* = b$. (A clarification: "there exists" is the "if" part of the question, while "unique" is the "only if" part.)

c) The primal problem

$$\max_{\lambda\in\mathbb{R}^m} \min_{x\in\mathbb{R}^n} \mathcal{L}(x,\lambda)$$

is equivalent to the original optimisation problem, which, from problem 2 in exercise set 8, has solution

$$x^* = A^\top(AA^\top)^{-1}b$$

As such, the optimal value of the primal problem becomes

$$\begin{aligned} \tfrac{1}{2}\|x^*\|^2 &= \tfrac{1}{2}b^\top(AA^\top)^{-\top}A\,A^\top(AA^\top)^{-1}b \\ &= \tfrac{1}{2}b^\top(AA^\top)^{-1}b, \end{aligned}$$

because $(AA^\top)^{-\top} = \big((AA^\top)^\top\big)^{-1} = (AA^\top)^{-1}$. From b) we have $\lambda^* = (AA^\top)^{-1}b$, so the optimal value of the dual problem also equals

$$q(\lambda^*) = \tfrac{1}{2}b^\top(AA^\top)^{-1}b,$$

after performing similar cancellations. Note that $x^* = A^\top\lambda^*$.

2 Introducing the Lagrangian

$$\begin{aligned}\mathcal{L}(x,\lambda,s) &= c^\top x - \lambda^\top(Ax-b) - s^\top x \\ &= b^\top\lambda + (c - A^\top\lambda - s)^\top x,\end{aligned}$$

the dual problem is defined as

$$\max_{\lambda\geq 0, s\geq 0} \min_x \mathcal{L}(x,\lambda,s). \tag{$\star$}$$

Since

$$\min_x \mathcal{L}(x,\lambda,s) = \begin{cases} -\infty & \text{if } A^\top\lambda + s \neq c; \\ b^\top\lambda & \text{if } A^\top\lambda + s = c,\end{cases}$$

we see that ($\star$) is equivalent to the problem

$$\max_{\lambda\geq 0, s\geq 0} b^\top\lambda \qquad \text{subject to} \qquad A^\top\lambda + s = c.$$

Interpreting s as a slack variable, we can further simplify the dual problem to

$$\max_\lambda b^\top\lambda \qquad \text{subject to} \qquad A^\top\lambda \leq c,\ \lambda \geq 0.$$

3 Abstractly, the linear optimisation problem may be written as

$$\min_x c^\top x \qquad \text{subject to} \qquad Ax = b,\ x \geq 0,$$

and its dual is, similarly as in the previous exercise,

$$\max_\lambda b^\top\lambda \qquad \text{subject to} \qquad A^\top\lambda \leq c.$$

Note that the constraint $\lambda \geq 0$ is not present in this case (why?). With

$$c = (5,3,4), \quad A = \begin{bmatrix}1 & 1 & 1\end{bmatrix}, \quad \text{and} \quad b = 1,$$

the actual dual becomes

$$\max \lambda \qquad \text{subject to} \qquad \lambda \leq 5,\ \lambda \leq 3,\ \lambda \leq 4,$$

with obvious solution $\lambda^* = 3$. Observe also that this gives us a convenient way of solving the primal problem: since only the second constraint is active, we have $x_1 = 0 = x_3$, and so the equality constraint of the primal problem yields that $x_2 = 1$.

4 **a)** We can see that the problem is a quadratic minimization problem

$$\min \frac{1}{2}x^\top Gx + c^\top x \quad \text{s.t.} \quad a_i^\top x - b_i \geq 0,$$

where

$$G = \begin{bmatrix} 8 & 2 \\ 2 & 2 \end{bmatrix}, \quad c = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad a_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad a_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

and $b_1 = 0$, $b_2 = -4$ and $b_3 = -3$. We can check that G is positive definite, so by Theorem 16.4 in N&W, the KKT conditions are necessary and sufficient for minimizers. We therefore set up the KKT conditions:

$$\begin{aligned} 8x + 2y + 2 - \lambda_1 + \lambda_2 + \lambda_3 &= 0 && \text{(1a)} \\ 2x + 2y + 3 + \lambda_1 + \lambda_3 &= 0 && \text{(1b)} \\ \lambda_1(x - y) &= 0 && \text{(1c)} \\ \lambda_2(4 - x - y) &= 0 && \text{(1d)} \\ \lambda_3(3 - x) &= 0 && \text{(1e)} \\ x - y &\geq 0 && \text{(1f)} \\ 4 - x - y &\geq 0 && \text{(1g)} \\ 3 - x &\geq 0. && \text{(1h)} \end{aligned}$$

We see from (1a) and (1b) that

$$\begin{aligned} x &= \frac{1}{6} + \frac{1}{3}\lambda_1 - \frac{1}{6}\lambda_3 \\ y &= -\frac{5}{3} - \frac{5}{6}\lambda_1 - \frac{1}{2}\lambda_2 - \frac{1}{6}\lambda_3. \end{aligned}$$

Now, we can go through the usual procedure of considering all options for active constraints. With no active constraints, i.e. $\lambda_1 = \lambda_2 = \lambda_3 = 0$, we get $(x, y) = (\frac{1}{6}, -\frac{5}{3})$ which is, in fact, a KKT point and as such a global solution of the problem. We should end the search for a minimum here, since the problem is strictly convex and the minimizer is unique, as confirmed by Figure 1. However, since the following discussion will prove useful in part b), we carry on looking for KKT points.

Next, we consider cases where only one constraint is active.

First, if $\lambda_1 = \lambda_2 = 0$, i.e. $3 - x = 0$, we get $\lambda_3 = -17$ and $(x, y) = (-\frac{16}{6}, -\frac{27}{16})$, which breaks constraint (1g).
Next, if $\lambda_1 = \lambda_3 = 0$, i.e. $4 - x - y = 0$, we get $\lambda_2 = -11$ and $(x, y) = (\frac{1}{6}, \frac{23}{6})$, which breaks constraint (1f).
Lastly, if $\lambda_2 = \lambda_3 = 0$, i.e. $x - y = 0$, we get $\lambda_1 = -\frac{11}{7}$ and $(x, y) = (-\frac{15}{42}, -\frac{15}{42})$. It is a feasible point, but has a negative Lagrange multiplier, meaning it is a candidate for a maximizer. This will prove useful in part b).

Next, we consider cases where two constraints are active.

First, if $\lambda_1 = 0$, i.e. $3 - x = 0$ and $4 - x - y = 0$, we get $(x, y) = (3, 1)$

with corresponding Lagrange multipliers $\lambda_2 = \frac{1}{3}$ and $\lambda_3 = -17$, meaning it is not a KKT point.

Next, if $\lambda_2 = 0$, i.e. $3 - x = 0$ and $x - y = 0$, we get $(x, y) =$ and $(x, y) = (3, 3)$, which breaks constraint (1g).

Lastly, if $\lambda_3 = 0$, i.e. $x - y = 0$ and $4 - x - y = 0$, we get $(x, y) = (2, 2)$ with corresponding Lagrange multipliers $\lambda_1 = \frac{11}{2}$ and $\lambda_2 = -\frac{33}{2}$, meaning it is not a KKT point.

There are no points in which all three constraints are active. Thus, we have one candidate for a minimizer, $(x, y) = (\frac{1}{6}, -\frac{5}{3})$, which is the global minimizer. Figure 1 shows the feasible domain and the contour lines of the objective function which confirm our observations.

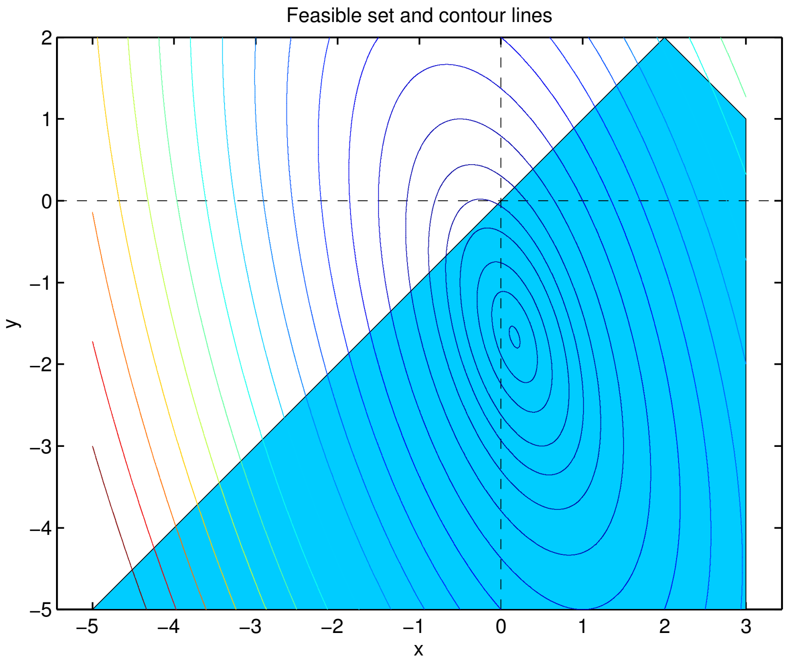

Figure 1: Feasible set (light blue) and contour lines of the function. Note: The feasible set extends further toward infinity.

b) Replacing f by $-f$ will turn minima into maxima and vice versa. Especially of note is that since $f \to \infty$ as $x^2 + y^2 \to \infty$, then $-f \to -\infty$, meaning there is no global solution to the minimization problem. However, the maximizer we found in the last problem, $(x, y) = (-\frac{15}{42}, -\frac{15}{42})$, now becomes local minimizer.