Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4180
Optimisation I
Spring 2017

Solutions to exercise set 6

1 **a)** We first define constraint functions

$$c_1(x, y) = y - x \quad \text{and} \quad c_2(x, y) = x^3 - y^4,$$

so that $\Omega = \{(x, y) \in \mathbb{R}^2 : c_1(x, y) \geq 0 \text{ and } c_2(x, y) \geq 0\}$, and sketch the region in Figure 1 below.

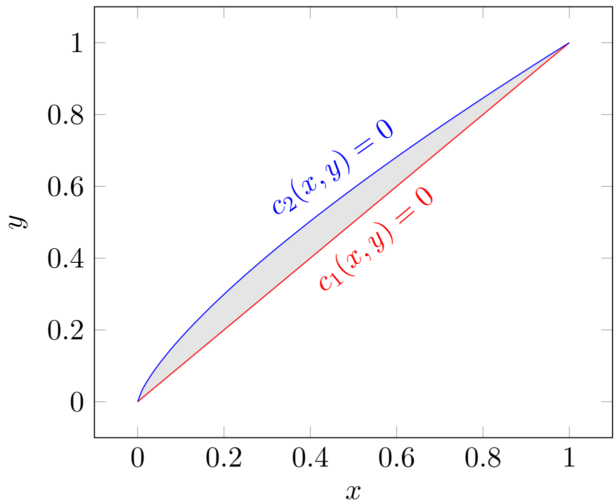

Figure 1: Region Ω in grey, with colors on the boundary specifiying the active constraints.

In order to characterise the tangent cone $T_\Omega(x, y)$ and the set of linearised feasible directions $\mathcal{F}(x, y)$, we employ Lemma 12.2 in N&W, which states that if the LICQ condition holds at a feasible point (x, y), then $T_\Omega(x, y) = \mathcal{F}(x, y)$.
Note first that the LICQ condition holds vacuously in the interior of Ω because all constraints are inactive, and therefore, $T_\Omega(x, y) = \mathcal{F}(x, y) = \mathbb{R}^2$ (why?) at interior points.
Next we consider boundary points with precisely one active constraint. Starting with points for which $c_1(x, y) = 0$—and excluding $(0, 0)$ and $(1, 1)$ where also c_2 is active—we find that $\nabla c_1(x, y) = (-1, 1)$. Since $\nabla c_1 \neq 0$, the LICQ condition holds, and so

$$\begin{aligned} T_\Omega(x, y) = \mathcal{F}(x, y) &= \left\{d \in \mathbb{R}^2 : \nabla c_1(x, y)^\top d \geq 0\right\} \\ &= \left\{d \in \mathbb{R}^2 : d_2 \geq d_1\right\}, \end{aligned}$$

where d is short for (d_1, d_2).

Similarly, if only c_2 is active, we observe that the LICQ condition holds because $\nabla c_2(x,y) = (3x^2, -4y^3) \neq 0$ away from $(0,0)$. This yields

$$T_\Omega(x,y) = \mathcal{F}(x,y) = \left\{d \in \mathbb{R}^2 : \nabla c_2(x,y)^\top d \geq 0\right\}$$
$$= \left\{d \in \mathbb{R}^2 : 3x^2 d_1 \geq 4y^3 d_2\right\}.$$

Constraint gradients at $(1,1)$ equal $\nabla c_1 = (-1,1)$ and $\nabla c_2 = (3,-4)$, which are linearly independent. Thus the LICQ condition is true, and

$$T_\Omega(1,1) = \mathcal{F}(1,1) = \left\{d \in \mathbb{R}^2 : \nabla c_1(1,1)^\top d \geq 0 \text{ and } \nabla c_2(1,1)^\top d \geq 0\right\}$$
$$= \left\{d \in \mathbb{R}^2 : 3d_1 \geq 4d_2\right\}.$$

Lastly, since $\nabla c_1(0,0) = (-1,1)$ and $\nabla c_2(0,0) = 0$, the LICQ condition fails at $(0,0)$, and we cannot expect that $T_\Omega(0,0) = \mathcal{F}(0,0)$. Readily,

$$\mathcal{F}(0,0) = \left\{d \in \mathbb{R}^2 : \nabla c_1(0,0)^\top d \geq 0 \text{ and } \nabla c_2(0,0)^\top d \geq 0\right\}$$
$$= \left\{d \in \mathbb{R}^2 : d_2 \geq d_1\right\}.$$

In order to find the tangent cone, we first consider limiting directions along the constraint boundaries $c_1(x,y) = 0$ and $c_2(x,y) = 0$ as $(x,y) \to (0,0)$. Travelling towards $(0,0)$ when c_1 is active, we may put, using the notation in N&W,

$$z_k = (1/k, 1/k) \quad \text{and} \quad t_k = 1/k,$$

and obtain the limiting direction

$$d = \lim_{k\to\infty} \frac{z_k - (0,0)}{t_k} = (1,1).$$

Note: the length of d is irrelevant; we only care about its direction. Similarly, travelling along $c_2(x,y) = 0$ yields $d = (0,1)$, using for example, the sequences

$$z_k = \left(k^{-1/3}, k^{-1/4}\right) \quad \text{and} \quad t_k = k^{-1/4}.$$

It can furthermore be seen that approaching $(0,0)$ from the interior of Ω gives tangent directions "between" these borderline cases, and so

$$T_\Omega(0,0) = \left\{d \in \mathbb{R}^2 : d_2 \geq d_1 \geq 0\right\}.$$

b) Defining

$$c_1(x,y) = y - x^4 \quad \text{and} \quad c_2(x,y) = x^3 - y$$

gives $\Omega = \{(x,y) \in \mathbb{R}^2 : c_1(x,y) \geq 0 \text{ and } c_2(x,y) \geq 0\}$, which is shown in Figure 2.

Omitting details—the process is very similar to the previous question—we obtain that the LICQ condition holds at all feasible points except $(0,0)$. Moreover, $T_\Omega(x,y) = \mathcal{F}(x,y)$ if (x,y) lies in the interior of Ω;

$$T_\Omega(x,y) = \mathcal{F}(x,y) = \left\{d \in \mathbb{R}^2 : d_2 \geq 4x^3 d_1\right\}$$

when only c_1 is active;

$$T_\Omega(x,y) = \mathcal{F}(x,y) = \left\{d \in \mathbb{R}^2 : 3x^2 d_1 \geq d_2\right\}$$

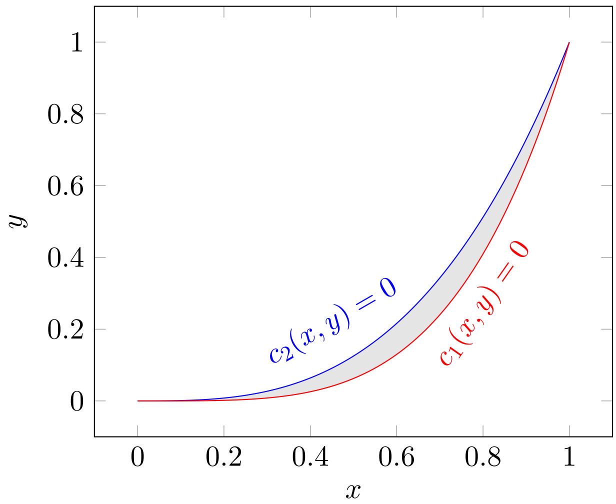

Figure 2: Region Ω in grey, with colors on the boundary specifiying the active constraints.

when only c_2 is active;

$$T_\Omega(1,1) = \mathcal{F}(1,1) = \left\{d \in \mathbb{R}^2 : 3d_1 \geq d_2 \geq 4d_1\right\};$$

and

$$\mathcal{F}(0,0) = \left\{d \in \mathbb{R}^2 : d_2 = 0\right\} \quad \text{and} \quad T_\Omega(0,0) = \left\{d \in \mathbb{R}^2 : d_2 = 0 \text{ and } d_1 \geq 0\right\}.$$

2 Let

$$\mathcal{L}(x, \lambda) = f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x)$$

be the Lagrangian associated with the maximisation problem. Since solving $\max_x f(x)$ is equivalent to solving $\min_x -f(x)$, we can state the KKT conditions for the minimisation problem. To this end, let

$$\widehat{\mathcal{L}}(x, \mu) = -f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \mu_i c_i(x)$$

be the Lagrangian for the minimisation problem, so that the KKT conditions become

$$\begin{aligned}
-\nabla f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \mu_i \nabla c_i(x) = \nabla_x \widehat{\mathcal{L}}(x, \mu) &= 0, \\
c_i(x) &= 0 \quad \text{for all } i \in \mathcal{E}, \\
c_i(x) &\geq 0 \quad \text{for all } i \in \mathcal{I}, \\
\mu_i &\geq 0 \quad \text{for all } i \in \mathcal{I}, \\
\mu_i c_i(x) &= 0 \quad \text{for all } i \in \mathcal{E} \cup \mathcal{I}.
\end{aligned}$$

Since

$$\mathcal{L}(x, -\mu) = -\widehat{\mathcal{L}}(x, \mu) \quad \text{and} \quad \nabla_x \mathcal{L}(x, -\mu) = -\nabla_x \widehat{\mathcal{L}}(x, \mu),$$

we see that changing the signs of the Lagrange multipliers, that is, putting $\lambda = -\mu$, is the only modification in the KKT conditions for the maximisation problem.

3 **a)** We begin by stating the problem in standard form, writing $\mathbf{x} = [x, y]^T$:

$$\min_{\mathbf{x}\in\mathbb{R}^2} f(\mathbf{x}) \quad \text{s.t.} \quad c_i(\mathbf{x}) \geq 0, \quad i = 1, 2, 3,$$

where

$$\begin{aligned} f(\mathbf{x}) &= x^2 + y^2, \\ c_1(\mathbf{x}) &= x + y - 1, \\ c_2(\mathbf{x}) &= 2 - y, \\ c_3(\mathbf{x}) &= y^2 - x. \end{aligned}$$

We then find the Lagrangian function and its gradient:

$$\mathcal{L}(\mathbf{x}, \lambda) = x^2 + y^2 - \lambda_1(x + y - 1) - \lambda_2(2 - y) - \lambda_3(y^2 - x)$$

$$\nabla_x \mathcal{L}(\mathbf{x}, \lambda) = \begin{bmatrix} 2x - \lambda_1 + \lambda_3 \\ 2y - \lambda_1 + \lambda_2 - 2y\lambda_3 \end{bmatrix}.$$

The KKT conditions can now be stated in full as:

$$2x^* - \lambda_1^* + \lambda_3^* = 0 \tag{1a}$$
$$2y^* - \lambda_1^* + \lambda_2^* - 2y^*\lambda_3^* = 0 \tag{1b}$$
$$x^* + y^* - 1 \geq 0 \tag{1c}$$
$$2 - y^* \geq 0 \tag{1d}$$
$$y^{*2} - x^* \geq 0 \tag{1e}$$
$$\lambda_i^* \geq 0, \quad i = 1, 2, 3 \tag{1f}$$
$$\lambda_1^*(x^* + y^* - 1) = 0 \tag{1g}$$
$$\lambda_2^*(2 - y^*) = 0 \tag{1h}$$
$$\lambda_3^*(y^{*2} - x^*) = 0. \tag{1i}$$

b) The feasible set is sketched in Figure 3.

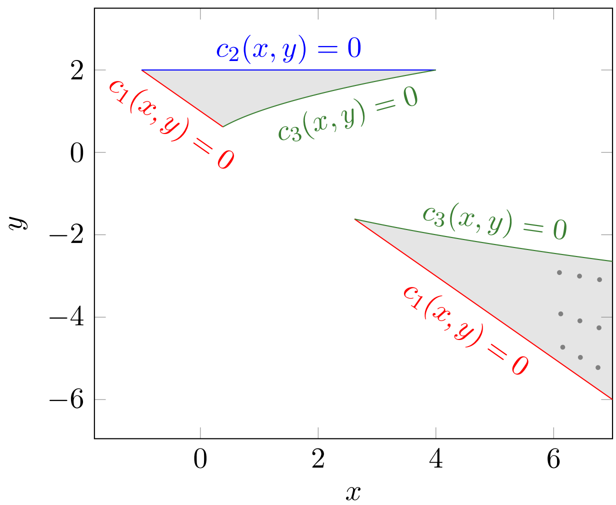

Figure 3: Feasible set. Note: The lower "triangle" extends further toward infinity.

We will find all KKT points by systematically considering all possible active sets of constraints. Remember that a constraint c_i is active at a point $\mathbf{x}$ if $c_i(\mathbf{x}) = 0$,

and that if all λ_i^* are negative at a point, then it is a candidate for a maximizer. Also, the LICQ conditions are satisfied at every point we consider here; with one active constraint, the LICQ conditions hold trivially, and in the cases with two constraints it is not hard to check that the LICQ conditions do hold.
Observe that if $\mathbf{x}^* = [x^*, y^*]^T$ is a KKT point, then from (1a) and (1b) we have:

$$x^* = \frac{\lambda_1^* - \lambda_3^*}{2}, \quad y^* = \frac{\lambda_1^* - \lambda_2^*}{2(1 - \lambda_3^*)}.$$

From here on, we will drop the asterisk in the notation and write x for x^*, etc.

First, if the active set is empty, i.e. neither of (1c)-(1e) are equalities. This corresponds to the interior of the domain. Then, by (1g)-(1i), we have $\lambda_1 = \lambda_2 = \lambda_3 = 0$, and so $x = y = 0$. But this point is not feasible, since it violates condition (1c). Thus, with the active set empty, there are no KKT points.

Next, we consider the case when the active set contains one index, i.e. exactly one of (1c)-(1e) is an equality. This corresponds to the boundaries of the domain, excepting the corner points. If (1c) is active, then $\lambda_2 = \lambda_3 = 0$ while $\lambda_1 \geq 0$. We get

$$x = \frac{\lambda_1}{2}, \quad y = \frac{\lambda_1}{2},$$

and inserting this into (1c) (which is now an equality), we get the condition

$$\frac{\lambda_1}{2} + \frac{\lambda_1}{2} - 1 = 0 \Rightarrow \lambda_1 = 1,$$

giving us the point $(x, y) = (\frac{1}{2}, \frac{1}{2})$. But this point violates condition (1e), so $(\frac{1}{2}, \frac{1}{2})$ is not a KKT point.

If (1d) is active, then $\lambda_1 = \lambda_3 = 0$ while $\lambda_2 \geq 0$, so

$$x = 0, \quad y = -\frac{\lambda_2}{2}.$$

Inserting this into the equality (1d), we get

$$2 + \frac{\lambda_2}{2} = 0 \Rightarrow \lambda_2 = -4.$$

Thus, (0,2) is a candidate for a maximizer. One can then check to verify that all KKT conditions are satisfied, and we find (0,2) to be a KKT point corresponding to a maximizer.

If (1e) is active, then $\lambda_1 = \lambda_2 = 0$ while $\lambda_3 \geq 0$, so

$$x = -\frac{\lambda_3}{2}, \quad y = 0.$$

Inserting this into the equality (1e), we get

$$\frac{\lambda_3}{2} = 0 \Rightarrow \lambda_3 = 0.$$

This gives the candidate point $(0, 0)$, which is not feasible since it violates (1c), and thereby is not a KKT point.

Having considered all possible active sets of one index, we now turn to the cases with two indices, i.e. exactly two of (1c)-(1e) are equalities. This corresponds to the corner points of the domain. First, if (1c) and (1d) are both active, then $\lambda_3 = 0$ while $\lambda_1, \lambda_2 \geq 0$. This gives us

$$x = \frac{\lambda_1}{2}, \quad y = \frac{\lambda_1 - \lambda_2}{2}.$$

Plugging this into equalities (1c) and (1d) yields:

$$\frac{\lambda_1}{2} + \frac{\lambda_1 - \lambda_2}{2} - 1 = 0$$
$$2 - \frac{\lambda_1 - \lambda_2}{2} = 0,$$

with solutions $\lambda_1 = -2$ and $\lambda_2 = -6$, yielding the KKT point (-1,2). Note that this is a candidate for a local maximizer, since all multipliers are negative.

Next, if (1c) and (1e) are both active, then $\lambda_2 = 0$ while $\lambda_1, \lambda_3 \geq 0$, which means

$$x = \frac{\lambda_1 - \lambda_3}{2}, \quad y = \frac{\lambda_1}{2(1 - \lambda_3)}.$$

Plugging this into equalities (1c) and (1e) yields:

$$\frac{\lambda_1 - \lambda_3}{2} + \frac{\lambda_1}{2(1 - \lambda_3)} - 1 = 0$$
$$\frac{\lambda_1^2}{4(1 - \lambda_3)^2} - \frac{\lambda_1 - \lambda_3}{2} = 0.$$

Solving this set of equations yields $\lambda_1 = 5 \pm \frac{9}{\sqrt{5}}$ and $\lambda_3 = 2 \pm \frac{4}{\sqrt{5}}$, thereby giving the candidate points $(x, y) = (\frac{1}{2}(3 \pm \sqrt{5}), \frac{1}{2}(-1 \mp \sqrt{5}))$ which both satisfy the KKT conditions. Since $\lambda_1, \lambda_3 \geq 0$, these points are minimizer candidates. Note: This result can be arrived upon by the easier approach of first finding the points (x, y) where c_1 and c_3 are both active, then working out what λ_1 and λ_3 are.

Finally, we check the case where (1d) and (1e) are both active, i.e. $\lambda_1 = 0$ while $\lambda_2, \lambda_3 \geq 0$. This gives us

$$x = -\frac{\lambda_3}{2}, \quad y = -\frac{\lambda_2}{2(1 - \lambda_3)}.$$

Plugging this into equalities (1d) and (1e) yields:

$$2 + \frac{\lambda_2}{2(1 - \lambda_3)} = 0$$
$$\frac{\lambda_2^2}{4(1 - \lambda_3)^2} + \frac{\lambda_3}{2} = 0,$$

which can be solved to find $\lambda_2 = -28$ and $\lambda_3 = -8$, giving the candidate point $(x, y) = (4, 2)$, which is a candidate for a maximizer, since the multipliers are negative.

Concerning the case with all constraints active, we may conclude that no KKT point exists; all three constraint functions cannot be active at the same point. The KKT points and their corresponding multipliers are summarized in the table below.

Point	λ_1	λ_2	λ_3	Minimizer/maximizer candidate
(0,2)	0	-4	0	Maximizer
$(\frac{1}{2}(3+\sqrt{5}), \frac{1}{2}(-1-\sqrt{5}))$	$5+\frac{9}{\sqrt{5}}$	0	$2+\frac{4}{\sqrt{5}}$	Minimizer
$(\frac{1}{2}(3-\sqrt{5}), \frac{1}{2}(-1+\sqrt{5}))$	$5-\frac{9}{\sqrt{5}}$	0	$2-\frac{4}{\sqrt{5}}$	Minimizer
(-1,2)	-2	-6	0	Maximizer
(4,2)	0	-28	-8	Maximizer

c) To determine whether the KKT points that are minimizer candidates are in fact local minimizers, we check the second order sufficient conditions from Theorem 12.6 in N&W, i.e. whether

$$w^T \nabla^2_{xx} \mathcal{L}(x, \lambda) w > 0 \ \forall w \in \mathcal{C}(x, \lambda), w \neq 0, \tag{2}$$

where, $\mathcal{C}(x, \lambda)$ is the critical cone at x, given by (12.53) in N&W.

For both candidates, i.e. $(\frac{1}{2}(3 \pm \sqrt{5}), \frac{1}{2}(-1 \mp \sqrt{5}))$, we have that the critical cone is simply given as $\mathcal{C}(x, \lambda) = \{0\}$. This is because any $w \in \mathcal{C}(x, \lambda)$ must be orthogonal to the $\nabla c_i(x)$ for which $\lambda_i > 0$, of which there are two for each point. Since the LICQ conditions hold at both points, these two vectors are linearily independent and thus span $\mathbb{R}^2$. The only vector orthogonal to $\mathbb{R}^2$ is the zero vector. Thereby, the only vector in $\mathcal{C}(x, \lambda)$ is the zero vector for these points, and thus condition (2) holds by default. We can conclude that $(\frac{1}{2}(3 \pm \sqrt{5}), \frac{1}{2}(-1 \mp \sqrt{5}))$ are strict local minimizers.

We note that $f(\frac{1}{2}(3-\sqrt{5}), \frac{1}{2}(-1+\sqrt{5})) < f(\frac{1}{2}(3+\sqrt{5}), \frac{1}{2}(-1-\sqrt{5}))$ and $f(\mathbf{x}) \to \infty$ in the unbounded region of the feasible domain. This means that $(\frac{1}{2}(3-\sqrt{5}), \frac{1}{2}(-1+\sqrt{5}))$ is a global minimizer and $(\frac{1}{2}(3+\sqrt{5}), \frac{1}{2}(-1-\sqrt{5}))$ is a local minimizer.

4 We begin by stating the problem in standard form, writing $\mathbf{x} = [x, y]^T$:

$$\min_{\mathbf{x} \in \mathbb{R}^2} f(\mathbf{x}) \quad \text{s.t.} \quad c_i(\mathbf{x}) \geq 0, \quad i = 1, 2$$

where

$$\begin{aligned} f(\mathbf{x}) &= x, \\ c_1(\mathbf{x}) &= y - x^4 \\ c_2(\mathbf{x}) &= x^3 - y \end{aligned}$$

The feasible set is sketched in Figure 2. We then find the Lagrange function and its gradient:

$$\mathcal{L}(\mathbf{x}, \lambda) = x - \lambda_1(y - x^4) - \lambda_2(x^3 - y)$$
$$\nabla_x \mathcal{L}(\mathbf{x}, \lambda) = \begin{bmatrix} 1 + 4x^3\lambda_1 - 3x^2\lambda_2 \\ -\lambda_1 + \lambda_2 \end{bmatrix},$$

and state the KKT conditions as:

$$1 + 4x^3\lambda_1 - 3x^2\lambda_2 = 0 \tag{3a}$$
$$-\lambda_1 + \lambda_2 = 0 \tag{3b}$$
$$y - x^4 \geq 0 \tag{3c}$$
$$x^3 - y \geq 0 \tag{3d}$$
$$\lambda_i \geq 0, \quad i = 1, 2 \tag{3e}$$
$$\lambda_1(y - x^4) = 0 \tag{3f}$$
$$\lambda_2(x^3 - y) = 0. \tag{3g}$$

Now, we can take a shortcut; from (3b), we see that $\lambda_1 = \lambda_2$, and from (3a) we see that there cannot exist any KKT point for which $\lambda_1 = \lambda_2 = 0$. Therefore, the cases with no active constraints ($\lambda_1 = \lambda_2 = 0$) and one active constraint ($\lambda_1 = 0$ or $\lambda_2 = 0$) cannot produce KKT points. We are left with considering the case where both constraints are active, i.e. the corner points (0,0) and (1,1).

In the point (1,1), we find (by (3a) and (3b)) that $\lambda_1 = \lambda_2 = -1$, corresponding to a candidate for a maximizer. In fact, the LICQ holds here, with two linearly independent ∇c_i, and know from the discussion in the previous exercise (two linearly independent vectors span $\mathbb{R}^2$ so the critical cone contains only the zero vector) that this is a local maximizer.

The last point is (0,0), for which we cannot write the gradient of f at (0,0) (which is $[1, 0]^T$) as a non-negative linear combination of the gradients of the constraints, and which therefore is not a KKT point. This does not, however, mean that it is not a minimizer. Applying common sense, it is clearly a local minimum, as no other points with $x = 0$ are feasible, and $x = 0$ is the lowest possible value of the objective function.

5 **a)** We begin, as usual, by stating the problem in standard form, writing $\mathbf{x} = [x, y]^T$:

$$\min_{\mathbf{x} \in \mathbb{R}^2} f(\mathbf{x}) \quad \text{s.t.} \quad c_i(\mathbf{x}) \geq 0, \quad i = 1, 2$$

where

$$f(\mathbf{x}) = xy,$$
$$c_1(\mathbf{x}) = y - x$$
$$c_2(\mathbf{x}) = x^3 - y^4$$

The feasible set is sketched in Figure 1. We find the Lagrange function and its gradient:

$$\mathcal{L}(\mathbf{x}, \lambda) = xy - \lambda_1(y - x) - \lambda_2(x^3 - y^4)$$
$$\nabla_x \mathcal{L}(\mathbf{x}, \lambda) = \begin{bmatrix} y + \lambda_1 - 3x^2\lambda_2 \\ x - \lambda_1 + 4y^3\lambda_2 \end{bmatrix},$$

and state the KKT conditions as:

$$y + \lambda_1 - 3x^2\lambda_2 = 0 \tag{4a}$$
$$x - \lambda_1 + 4y^3\lambda_2 = 0 \tag{4b}$$
$$y - x \geq 0 \tag{4c}$$
$$x^3 - y^4 \geq 0 \tag{4d}$$
$$\lambda_i \geq 0, \quad i = 1, 2 \tag{4e}$$
$$\lambda_1(y - x) = 0 \tag{4f}$$
$$\lambda_2(x^3 - y^4) = 0. \tag{4g}$$

Now, we can check the different cases of active constraints to find KKT points. First, with no active constraints, i.e. $\lambda_1 = \lambda_2 = 0$, we get the point (0,0). In fact, this is a point with both constraints active; it just so happens that $\lambda_1 = \lambda_2 = 0$ here. One can check that the LICQ does not hold here, but it is still a KKT point because the gradient of f at (0,0) (which is 0) can be written as a non-negative linear combination of the gradients of the constraints. Thereby, we conclude that there exist no KKT points with no active constraints, but that (0,0) is a KKT point. What the failure of the LICQ at (0,0) implies is the following: There exists a function f which has a local minimum at (0,0), but for which (0,0) is not a KKT point. However, we might still be lucky for any given function f, as is the case here.

Next, we check with one active constraint. First, with $\lambda_1 \geq 0, \lambda_2 = 0$, we have from (4a) and (4b) that $x = \lambda_1$ and $y = -\lambda_1$. Inserting into the equality (4c) yields $\lambda_1 = 0$, and therefore $(x, y) = (0, 0)$ again, which has been discussed already.

With $\lambda_2 \geq 0, \lambda_1 = 0$, equations (4a), (4b) and (4d) become

$$y - 3x^2\lambda_2 = 0,$$
$$x + 4y^3\lambda_2 = 0,$$
$$x^3 = y^4.$$

Multiplying the first of these by x, the second by y, applying the third and adding the two first gives

$$y^4\lambda_2 = 0.$$

Any solution of this leads to the point (0,0), which we have already found to be a KKT point.

Finally, we check the case with two active constraints, for which there are two points; (0,0), which is already considered, and (1,1). In the point (1,1), we find (by (4a) and (4b)) that $\lambda_1 = -7$ and $\lambda_2 = -2$. All other KKT conditions are satisfied, so (1,1) is a KKT point candidate for a maximizer.

The only minimizer candidate we have is (0,0), for which the LICQ did not hold, and which therefore can be neither confirmed or discarded as a minimizer/maximizer using the second order necessary and sufficient conditions. However, it is clearly a local (and even global) minimizer, as we are only considering nonnegative values for x and y, and since $f(x, y) = xy$, its global minimum (in the feasible set) is located at (0,0).

b) To find the critical cone C at $(0, 0)$, we use the definition given by equation (12.53) in N&W page 330. First, we find the gradients of the constraints at this point:

$$\nabla c_1(0,0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \nabla c_2(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $\lambda_1 = \lambda_2 = 0$ at this point, we have that $d \in C(0, 0)$ if and only if $\nabla c_1(0, 0)^T d \geq 0$ and $\nabla c_2(0, 0)^T d \geq 0$. The latter condition clearly holds for all d, and so we find that

$$\begin{aligned} C(0,0) &= \{d = (d_1, d_2) \in \mathbb{R}^2 : \nabla c_1(0,0)^T d \geq 0\} \\ &= \{d = (d_1, d_2) \in \mathbb{R}^2 : d_2 \geq d_1\}. \end{aligned}$$

Next, we find that the Hessian of the Lagrangian at $(0, 0)$ with Lagrange multipliers $\lambda^* = (\lambda_1, \lambda_2) = (0, 0)$ is given by

$$\nabla^2 \mathcal{L}((0,0); (\lambda_1, \lambda_2)) = \nabla^2 \mathcal{L}((0,0); (0,0)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus, $d^T \nabla^2 \mathcal{L}((0, 0); (\lambda_1, \lambda_2))d = 2d_1 d_2$. There are clearly directions in the critical cone for which this is negative; one can choose $d_2 > 0$ and $d_1 < 0$.

c) We can find the tangent cone to the feasible set at $(0, 0)$ by looking at the limiting vectors along the lines $c_1(\mathbf{x}) = 0$ and $c_2(\mathbf{x}) = 0$ as $\mathbf{x} \to 0$. Traveling toward (0,0) along $c_1(\mathbf{x}) = 0$ we take, for example,

$$z_k = \begin{bmatrix} 1/k \\ 1/k \end{bmatrix}, \quad t_k = ||z_k|| = \sqrt{2}/k,$$

and find the limiting direction

$$d = \lim_{k \to \infty} \frac{z_k}{t_k} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

Along $c_2(\mathbf{x}) = 0$, we take

$$z_k = \begin{bmatrix} 1/k \\ 1/k^{3/4} \end{bmatrix}, \quad t_k = ||z_k|| = \frac{\sqrt{\sqrt{k} + 1}}{k},$$

and find the limiting direction

$$d = \lim_{k \to \infty} \frac{z_k}{t_k} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The tangent cone at (0,0) contains all vectors between these limiting cases, which can be shown to be:

$$T(0,0) = \{d \in \mathbb{R}^2 : d_1 \geq 0 \text{ and } d_2 \geq d_1\}.$$

It is then easy to see that $d^T \nabla^2 \mathcal{L}((0,0);(\lambda_1, \lambda_2))d = 2d_1 d_2 \geq 0$ for all d in the tangent cone at $(0,0)$.