Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4180 Optimisation I Spring 2017

Exercise set 9

This exercise sheet deals both with linear and quadratic optimisation, but at the same time discusses, to some degree, duality results for these types of optimisation problems. To that end, we first recall the basic idea of duality for constrained optimisation as discussed in the lectures:

Assume that we are given a constrained optimisation problem of the form

$$\min_x f(x) \qquad \text{subject to } \begin{cases} c_i(x) = 0, & i \in \mathcal{E}, \\ c_i(x) \geq 0, & i \in \mathcal{I}. \end{cases} \tag{1}$$

Then its Lagrangian is the mapping $\mathcal{L} \colon \mathbb{R}^n \times \mathbb{R}^{\mathcal{E}\cup\mathcal{I}}$ defined as

$$\mathcal{L}(x, \lambda) = f(x) - \sum_{i \in \mathcal{E}\cup\mathcal{I}} \lambda_i c_i(x).$$

Now define the function $p \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$,

$$p(x) := \max_{\substack{\lambda \in \mathbb{R}^{\mathcal{E}\cup\mathcal{I}} \\ \lambda_i \geq 0,\ i \in \mathcal{I}}} \mathcal{L}(x, \lambda).$$

Then it is easy to see[1] that

$$p(x) = \begin{cases} f(x) & \text{if } c_i(x) = 0,\ i \in \mathcal{E}, \text{ and } c_i(x) \geq 0,\ i \in \mathcal{I}, \\ +\infty & \text{else.} \end{cases}$$

Thus[2] solving the constrained optimisation problem (1) is equivalent to solving the unconstrained problem

$$\min_{x \in \mathbb{R}^n} p(x),$$

or, explicitly, the *primal problem*

$$\min_{x \in \mathbb{R}^n} \max_{\substack{\lambda \in \mathbb{R}^{\mathcal{E}\cup\mathcal{I}} \\ \lambda_i \geq 0,\ i \in \mathcal{I}}} \mathcal{L}(x, \lambda). \tag{2}$$

Now one defines the *dual problem* by exchanging the order of the minimum and the maximum in (2). That is, the dual problem is

$$\max_{\substack{\lambda \in \mathbb{R}^{\mathcal{E}\cup\mathcal{I}} \\ \lambda_i \geq 0,\ i \in \mathcal{I}}} \min_{x \in \mathbb{R}^n} \mathcal{L}(x, \lambda). \tag{3}$$

[1] Read: Try to prove this result yourself!

[2] The same remark as in footnote 1 applies.

In other words, we first define a function $q\colon \mathbb{R}^{\mathcal{E}\cup\mathcal{I}} \to \mathbb{R} \cup \{-\infty\}$ by

$$q(\lambda) := \min_{x\in\mathbb{R}^n} \mathcal{L}(x,\lambda),$$

and then maximise q with respect to λ subject to the constraint that the Lagrange parameters for the inequality constraints are nonnegative. Note that the minimum in the definition of q is taken over *all* $x \in \mathbb{R}^n$ irrespective of the constraints.

(Also: If one wants to be accurate, one should always read the minima and maxima in the definitions of the primal and the dual problem as infima and suprema, respectively, as it is not clear that these optimisation problems actually have solutions.)

One can easily[3] show that the dual problem has the following properties:

- The function $q\colon \mathbb{R}^{\mathcal{E}\cup\mathcal{I}} \to \mathbb{R} \cup \{-\infty\}$ is concave (but possibly constant $-\infty$).
- The value of the dual problem is smaller than or equal to the value of the primal problem. That is,

$$\max_{\substack{\lambda\in\mathbb{R}^{\mathcal{E}\cup\mathcal{I}}\\ \lambda_i\ge 0,\ i\in\mathcal{I}}} \min_{x\in\mathbb{R}^n} \mathcal{L}(x,\lambda) \le \min_{x\in\mathbb{R}^n} \max_{\substack{\lambda\in\mathbb{R}^{\mathcal{E}\cup\mathcal{I}}\\ \lambda_i\ge 0,\ i\in\mathcal{I}}} \mathcal{L}(x,\lambda). \tag{4}$$

  This property is called *weak duality*.

If, for some specific constrained optimisation problem, we actually have equality in (4) and both the primal and the dual problem have solutions x^* and λ^*, then we say that *strong duality* holds, and we call the pair (x^*,λ^*) a *primal–dual solution* of the constrained optimisation problem we have started with. Under certain additional conditions, it is possible to show the existence of such primal–dual solutions. Also, in such a case the point x^* is a KKT point with corresponding Lagrange multiplier λ^*.

1 Assume that $A \in \mathbb{R}^{m\times n}$ with $m < n$ is a matrix of full rank and that $b \in \mathbb{R}^m \setminus \{0\}$. Consider the optimization problem

$$\frac{1}{2}\|x\|^2 \to \min \qquad \text{subject to} \qquad Ax = b. \tag{5}$$

(See also problem 2 in exercise set 8.)

a) Derive an explicit formula for the dual problem to (5).

b) Show that $\lambda^* \in \mathbb{R}^m$ solves the dual problem if and only if

$$AA^\top \lambda^* = b.$$

c) Verify that in this situation

$$\min_{x\in\mathbb{R}^n} \max_{\lambda\in\mathbb{R}^m} \mathcal{L}(x,\lambda) = \max_{\lambda\in\mathbb{R}^m} \min_{x\in\mathbb{R}^n} \mathcal{L}(x,\lambda).$$

[3] Again, see footnote 1. Alternatively, have a look at Theorems 12.10 and 12.11 in N&W.

2 Find—and simplify, if possible—the dual of the linear programme

$$\min c^\top x \qquad \text{subject to } Ax \geq b,\ x \geq 0.$$

3 Find the dual of the linear optimisation problem

$$5x_1 + 3x_2 + 4x_3 \to \min \qquad \text{subject to } \begin{cases} x_1 + x_2 + x_3 = 1, \\ \qquad\qquad x_i \geq 0, \quad i = 1, 2, 3, \end{cases}$$

and compute its (i.e., the *dual's*) solution.

4 (*See exercise 16.1 in Nocedal & Wright.*) Consider the quadratic programme

$$f(x, y) := 2x + 3y + 4x^2 + 2xy + y^2 \to \min$$

subject to

$$x - y \geq 0, \qquad x + y \leq 4, \qquad x \leq 3.$$

a) Solve the quadratic programme and sketch its geometry (that is, the domain of the problem and the level lines of the function f).

b) What happens if one replaces the function f by $-f$? Does the problem still have solutions or local solutions?