Norwegian University of Science and Technology
Department of Mathematical Sciences

**Exercise set 8**

**1** Consider the constrained optimisation problem

$$x + y \to \min \qquad \text{subject to } x^2 + y^2 \le 1.$$

Formulate a logarithmic barrier method for the solution of this constrained optimisation problem and compute its solution for each parameter $\mu > 0$ in the barrier functional.

**2** Assume that $A \in \mathbb{R}^{m\times n}$ with $m < n$ is a matrix of full rank and that $b \in \mathbb{R}^m \setminus \{0\}$. Consider the optimisation problem

$$\frac{1}{2}\|x\|^2 \to \min \qquad \text{subject to} \qquad Ax = b. \tag{1}$$

(See also problem 3 in exercise set 7.)

**a)** Formulate the augmented Lagrangian $\mathcal{L}_A$ for problem (1), and find for all possible parameters $\lambda \in \mathbb{R}^m$ and $\mu > 0$ a formula for the global solutions of the resulting unconstrained optimisation problem.

**b)** For which parameters $\lambda \in \mathbb{R}^m$ and $\mu > 0$ is the minimiser of the augmented Lagrangian equal to the solution of (1)?

**c)** An iterative algorithm for the solution of (1) using the augmented Lagrangian may have the form

$$\begin{aligned} x^{k+1} &\in \arg\min_x \mathcal{L}_A(x, \lambda^k; \mu), \\ \lambda^{k+1} &= \lambda^k - \mu(Ax^{k+1} - b). \end{aligned}$$

Show that this iteration converges for all initial values $x^0 \in \mathbb{R}^n$, $\lambda^0 \in \mathbb{R}^m$, and all $\mu > 0$ to the unique solution of (1).[1]

*Hint:* Interpret the iteration for the Lagrange parameter $\lambda$ as a fixed-point iteration—by using the explicit formula for $x^{k+1}$ derived in the first part of this exercise—and then use results from previous numerics courses to show that this fixed-point scheme converges.

*Useful matrix formula:* if $B$, $C$, and $B + C$ are invertible matrices of the same size, then

$$\left(B^{-1} + C^{-1}\right)^{-1} = B(B + C)^{-1}C = C(B + C)^{-1}B.$$

[1] This problem does not completely fall within the curriculum of this optimisation class, but it is still recommended to *try* to solve it.