Norwegian University of Science and Technology
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TMA4180 Optimization I Spring 2017

Exercise set 7

1 Consider the constrained optimization problem

$$-x^2 - (y-1)^2 \to \min \qquad \text{such that} \qquad \begin{cases} y \geq Cx^2, \\ y \leq 2, \end{cases}$$

where $C > 0$ is some positive parameter.

a) Show that the point $(0,0)$ is a KKT point for all parameters $C > 0$ and that the LICQ is satisfied at $(0,0)$.

b) Formulate the second order necessary and sufficient optimality conditions for the point $(0,0)$. For which parameters C are these conditions satisfied? For which parameters C is the point $(0,0)$ a local minimum?

2 Consider the constrained optimisation problem

$$\frac{1}{2}(x^2 + y^2) \to \min \qquad \text{subject to } xy = 1.$$

a) Find—by whatever means—the solutions of this problem. In addition, find the values of the corresponding Lagrange multipliers.

b) Formulate the unconstrained optimisation problem that results from the application of the quadratic penalty method with parameter $\mu > 0$. Solve these problems for all possible parameters μ and verify that the solutions converge to the solutions of the constrained optimization problem as $\mu \to \infty$.

c) Formulate the augmented Lagrangian for this constrained optimization problem and find (for all possible parameters $\lambda \in \mathbb{R}$ and $\mu > 0$) the global solutions of this (unconstrained) optimization problem. For which parameters does one recover the solution of the original constrained problem?

d) The ℓ^1-penalty function for this optimisation problem is defined, for some parameter $\mu > 1$, as

$$\Phi_1(x, y; \mu) := \frac{1}{2}(x^2 + y^2) + \mu|xy - 1|.$$

Find for each parameter $\mu > 0$ the global minimisers of this function. For which parameters $\mu > 0$ do they coincide with the solutions of the original problem?

3 Assume that $A \in \mathbb{R}^{m \times n}$ with $m < n$ is a matrix of full rank and that $b \in \mathbb{R}^m \setminus \{0\}$. Consider the optimization problem

$$\frac{1}{2}\|x\|^2 \to \min \qquad \text{subject to} \qquad Ax = b. \tag{1}$$

a) Formulate the KKT-conditions for this problem and show that the unique solution is given by

$$x^* = A^\top (AA^\top)^{-1} b.$$

b) Formulate the quadratic penalty method for this constrained optimization problem, and show that the unique minimizer with parameter $\mu > 0$ is given by

$$x_\mu := A^\top \Big(\frac{1}{\mu}\,\mathrm{Id} + AA^\top\Big)^{-1} b$$

with $\mathrm{Id} \in \mathbb{R}^{m \times m}$ denoting the identity matrix.

c) Now consider the optimization problem

$$\frac{1}{2}\|x\|^2 \to \min \qquad \text{subject to} \qquad \frac{1}{2}\|Ax - b\|^2 \le \varepsilon$$

for some $\varepsilon > 0$, and denote its solution by $\widehat{x}_\varepsilon$. Show that either $\frac{1}{2}\|b\|^2 \le \varepsilon$ (in which case $\widehat{x}_\varepsilon = 0$), or there exists $\mu > 0$ such that $\widehat{x}_\varepsilon = x_\mu$.