Norwegian University of Science and Technology
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TMA4180 Optimization I Spring 2017

Exercise set 6

The following problems are concerned with constrained optimisation and with the concepts of tangent cones and linearized feasible directions. Problem 3 is in some sense the "typical" case, where the KKT conditions work perfectly fine for finding the solutions. In contrast, problems 4 and 5 deal with slightly more difficult situations.

1 For the following two examples, sketch the region Ω defined by the constraints and compute for each point in Ω both the tangent cone and the set of linearized feasible directions. For which points is the LICQ satisfied?

a) The region $\Omega \subset \mathbb{R}^2$ defined by the inequalities

$$y \geq x \qquad \text{and} \qquad y^4 \leq x^3.$$

b) The region $\Omega \subset \mathbb{R}^2$ defined by the inequalities

$$y \geq x^4 \qquad \text{and} \qquad y \leq x^3.$$

Note that the sets Ω considered in this example occur again in problems 4 and 5 on this exercise sheet.

2 Assume that one wants to solve the optimisation problem

$$\max_x f(x) \qquad \text{such that} \qquad \begin{cases} c_i(x) = 0 & \text{for all } i \in \mathcal{E}, \\ c_i(x) \geq 0 & \text{for all } i \in \mathcal{I}. \end{cases}$$

How do the KKT conditions have to be modified such that one obtains (first order) necessary conditions for this maximisation problem?

3 Consider the constrained optimization problem

$$x^2 + y^2 \to \min \qquad \text{such that} \qquad \begin{cases} x + y \geq 1, \\ y \leq 2, \\ y^2 \geq x. \end{cases}$$

a) Formulate the KKT-conditions for this optimization problem.

b) Find all KKT points for this optimization problem.

c) Find all local and global minima for this optimization problem.

(Part b can be very tedious. One strategy is to consider all possible active sets and determine for each active set whether KKT-points exist. It can also be extremely helpful to sketch the feasible set and the function.)

4 Consider the constrained optimization problem

$$x \to \min \qquad \text{such that} \qquad \begin{cases} y \geq x^4, \\ y \leq x^3. \end{cases}$$

Find all KKT points and local minima for this optimization problem.

5 Consider the constrained optimization problem

$$xy \to \min \qquad \text{such that} \qquad \begin{cases} y \geq x, \\ y^4 \leq x^3. \end{cases}$$

a) Find all KKT points and local minima for this optimization problem.

b) Compute the critical cone at $(0,0)$ as defined in the lecture and Nocedal & Wright, and show that there exist directions d contained in the critical cone for which $d^\top \nabla^2 \mathcal{L}((0,0); \lambda^*) d < 0$.

c) Show that $d^\top \nabla^2 \mathcal{L}((0,0); \lambda^*) d \geq 0$ for all vectors d contained in the tangent cone to the feasible set at $(0,0)$.