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Exercise set 2

1 Consider the function

$$f(x, y, z) = 2x^2 + xy + y^2 + yz + z^2 - 6x - 7y - 8z + 9.$$

a) Using the first order necessary conditions, find a critical point of f.

b) Use the second-order sufficient conditions in order to verify that this point is a local minimum.

c) Prove that this point is a global minimum of f.

2 Assume that f is a continuously differentiable function satisfying

$$\lim_{\|x\|\to\infty} \frac{f(x)}{\|x\|} = +\infty.$$

Show that the equation

$$\nabla f(x) = u$$

has a solution for every $u \in \mathbb{R}^n$.

Hint: Consider global minima of the function $f_u(x) := f(x) - u^T x$.

3 Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x, y) = \log(e^x + e^y)$$

is convex.

4 (Arithmetic–Geometric Mean Inequality) Let α, $\beta > 0$ be such that $\alpha + \beta = 1$. Show that for all x, $y > 0$ one has

$$x^\alpha y^\beta \leq \alpha x + \beta y$$

with equality if and only if $x = y$.

Hint: show that the function $-\log x$ *is strictly convex on* $\mathbb{R}_{>0}$.

5 *(See N&W, Exercise 2.8)* Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function. Show that the set of minimisers of f is convex.

6 Show that a strictly convex function $f\colon \mathbb{R}^n \to \mathbb{R}$ has at most one global minimiser. In addition, find a strictly convex function that has no global minimiser at all.

7 Assume that $f\colon \mathbb{R}^n \to \mathbb{R}$ is a strictly convex and continuous function that attains a minimiser. Show that f is coercive.[1]

Hint: assuming x^* is the minimiser of f, show that

$$f(x) \geq f(x^*) + \big(c - f(x^*)\big)\|x - x^*\|$$

whenever $\|x - x^*\| \geq 1$, where $c > f(x^*)$ is the minimum of f on $\{x : \|x - x^*\| = 1\}$.

[1] The continuity of f is strictly speaking not a necessary requirement, as it can be shown that every convex function defined on the whole space $\mathbb{R}^n$ is necessarily continuous.