Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4180 Optimisation I Spring 2017

Exercise set 1

$\boxed{1}$ *(Properties of the lower limit)*

a) Let $(y_k)_{k\in\mathbb{N}}$, $(z_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ be two real sequences. Show that

$$\liminf_{k\to\infty} y_k + \liminf_{k\to\infty} z_k \le \liminf_{k\to\infty}(y_k + z_k).$$

In addition, find an example where this inequality is strict.

b) Let I be any index set (possibly infinite), and let $(y_k^i)_{k\in\mathbb{N}} \subset \mathbb{R}$, $i \in I$, be a family of sequences. Show that

$$\sup_{i\in I} \liminf_{k\to\infty} y_k^i \le \liminf_{k\to\infty}\bigl(\sup_{i\in I} y_k^i\bigr).$$

$\boxed{2}$ Let I be any index set and let $f_i \colon \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be lower semi-continuous. Show that the function $f\colon \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ given as

$$f(x) = \sup_{i\in I} f_i(x)$$

is lower semi-continuous.

$\boxed{3}$ For the following functions, decide whether they are lower semi-continuous or coercive, and whether they attain a global minimizer:

a) The function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x^4 - 20x^3 + \sup_{k\in\mathbb{N}} \sin(kx).$$

b) The function $g\colon \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = e^x - \frac{1}{x^2+1}.$$

c) The function $h\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$h(x) = x_1^2(1 + x_2^3) + x_1^2.$$

$\boxed{4}$ For which constants $c \in \mathbb{R}$ does the function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) := (x^2 - 2x + c)e^{-x^2}$$

attain a global minimum?

5 Given a matrix $A \in \mathbb{R}^{n \times n}$, we denote by

$$\|A\|_F := \Big(\sum_{i,j=1}^{n} a_{ij}^2\Big)^{1/2}$$

its *Frobenius norm*. Show that the optimisation problem

$$\min_{\substack{A \in \mathbb{R}^{n \times n} \\ \det A > 0}} \Big(\|A\|_F + \frac{1}{\det A}\Big)$$

admits a global minimum.

6 *(See N&W, Exercise 2.1).* The *Rosenbrock function* is defined as

$$f(x) := 100(x_2 - x_1^2)^2 + (1 - x_1)^2.$$

a) Compute the gradient and the Hessian of the Rosenbrock function.

b) Show that the point $(1, 1)$ is the unique (global and local) minimizer of f.