Problem 1 Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = y^4 + 3y^2 - 4xy - 2y + x^2.$$

a) Compute all stationary points of f, and find all local or global minima of f.
(10 points)

b) Decide whether the function f is convex or not.
(5 points)

c) Starting at the point $(x,y) = (0,0)$, compute one step of the gradient descent method with backtracking (Armijo) linesearch (see Algorithm 3.1 in Nocedal and Wright). Start with an initial step length $\bar{\alpha} = 1$, and use the parameters $c = 1/10$ (sufficient decrease parameter) and $\rho = 1/4$ (contraction factor).
(10 points)

Problem 2 Consider the constrained optimization problem

$$2x^2 - y^2 - 2y \to \min \qquad \text{subject to } x + y = 1.$$

Formulate the unconstrained optimization problem resulting from an application of the quadratic penaly method applied to this problem. Determine for which parameters $\mu > 0$ the resulting unconstrained problems has a solution, and compute the solution for all parameters for which it exists.
(10 points)

Problem 3 We consider the problem of solving the constrained optimization problem

$$(x-2)^2 + (y-1)^2 \to \min \qquad \text{subject to } (x,y) \in \Omega,$$

where the set $\Omega \subset \mathbb{R}^2$ is given by the inequality constraints

$$\begin{aligned} y &\ge 0, \\ 1 - x &\ge 0, \\ x^2 - y &\ge 0. \end{aligned}$$

a) Sketch the set Ω and find all points $(x,y) \in \Omega$ where the LICQ fails to hold.
(5 points)

b) Determine the tangent cone and the set of linearized feasible directions to the set Ω in the points $(x,y) = (1,1)$ and $(x,y) = (0,0)$.
(10 points)

c) Use the second order optimality conditions in order to show that the point $(1, 1)$ is a local solution of this optimization problem.
(15 points)

Problem 4 Find the dual of the linear optimization problem

$$x + y - z \to \min \qquad \text{subject to } \begin{cases} x - y - 3z = -1, \\ x,\ y,\ z \geq 0, \end{cases}$$

and compute the solutions of both the primal and the dual problem.
(10 points)

Problem 5 Assume that $C \subset \mathbb{R}^n$ is a closed and convex, non-empty set. Given any point $y \in \mathbb{R}^n$, we define the projection $P_C(y)$ of y to be the solution of the optimization problem

$$\min_x \frac{1}{2}(x - y)^2 \qquad \text{subject to } x \in C. \tag{1}$$

a) Show that the optimization problem (1) has a solution, and that this solution is unique.
(5 points)

We now assume that the set C is given by

$$x \in C \iff c(x) \geq 0,$$

where $c \colon \mathbb{R}^n \to \mathbb{R}$ is a concave (that is, $-c$ is convex) and smooth function.

b) Assume that there exists some $x \in C$ such that $c(x) > 0$. Show that the projection of a point y to the set C is characterized by the following conditions:

- If $y \in C$, then $P_C(y) = y$.
- If $y \notin C$, then $x = P_C(y)$, if and only if $x \in C$ and there exists $\lambda > 0$ such that
$$x - y = \lambda \nabla c(x).$$

(10 points)

We now consider the numerical solution of an optimization problem of the form

$$f(x) \to \min \qquad \text{subject to } x \in C, \tag{2}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}$ is a convex and smooth function.

One method for solving this problem is the so called *gradient projection method*, which is defined by the iteration

$$x_{k+1} = P_C\Big(x_k - \tau \nabla f(x_k)\Big), \tag{3}$$

where $\tau > 0$ is some fixed parameter.

c) Assume that the iteration x_{k+1} converges to some $x^* \in \mathbb{R}^n$. Show that x^* is a solution of (2).
(10 points)