Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4180 Optimization I Spring 2016

Exercise set 1

1 *(Properties of the lower limit)*

a) Let $(y_k)_{k\in\mathbb{N}}$, $(z_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ be two real sequences. Show that

$$\liminf_{k\to\infty} y_k + \liminf_{k\to\infty} z_k \leq \liminf_{k\to\infty}(y_k + z_k).$$

In addition, construct an example, where the inequality is strict.

b) Let I be any index set and let $(y_k^i)_{k\in\mathbb{N}} \subset \mathbb{R}$ be a family of sequences. Show that

$$\sup_{i\in I} \liminf_{k\to\infty} y_k^i \leq \liminf_{k\to\infty}\bigl(\sup_{i\in I} y_k^i\bigr).$$

2 Let I be any index set and let $f_i\colon \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be lower semi-continuous. Show that the function $f\colon \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ given as

$$f(x) = \sup_{i\in I} f_i(x)$$

is lower semi-continuous.

3 For the following functions, decide whether they are lower semi-continuous or coercive, and whether they attain a global minimizer:

a) The function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x^4 - 20x^3 + \sup_{k\in\mathbb{N}} \sin(kx).$$

b) The function $g\colon \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = e^x - \frac{1}{x^2+1}.$$

c) The function $h\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$h(x) = x_1^2(1 + x_2^3) + x_1^2.$$

4 *(See N&W, Exercise 2.8)* Assume that $f\colon \mathbb{R}^n \to \mathbb{R}$ is a convex function. Show that the set of minimizers of f is convex.

5 *(See N&W, Exercise 2.1).* The *Rosenbrock function* is defined as

$$f(x) := 100(x_2 - x_1^2)^2 + (1 - x_1)^2.$$

a) Compute the gradient and the Hessian of the Rosenbrock function.

b) Show that the point $(1, 1)$ is the unique (global and local) minimizer of f.

6 Show that a strictly convex function $f : \mathbb{R}^n \to \mathbb{R}$ has at most one global minimizer. In addition, find a strictly convex function that has no global minimizer at all.

7 Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is convex, continuous, and bounded below. Show that for every $\lambda > 0$ the function

$$f_\lambda(x) := f(x) + \lambda\|x\|^2$$

has a unique global minimizer.

(*Additional challenge: Is it possible to drop the condition that f is bounded below?*)