NTNU
Norwegian University of
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Department of Mathematical Sciences

Examination paper for **TMA4180 Optimization I**

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Examination date: 26th May 2016
Examination time (from–to): 09:00–13:00
Permitted examination support material:

- The textbook: Nocedal & Wright, Numerical Optimization including errata.
- Rottmann, Mathematical formulae.
- Handouts on *Minimisers of unconstrained optimisation problems*, *Basic properties of convex functions*.
- Approved basic calculator.

Other information:

- All answers should be justified and include enough details to make it clear which methods or results have been used.
- You may answer to the questions of the exam either in English or in Norwegian.

Language: English
Number of pages: 2
Number of pages enclosed: 0

Checked by:

Date Signature

Problem 1 Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = 2x^2 - 2xy^2 - 12x + y^4 + 2y^2 + 36.$$

a) Compute all stationary points of f and find all local or global minimizers of f.
(10 points)

b) Starting at the point $(x,y) = (1,2)$ compute one step of Newton's method with backtracking (Armijo) linesearch (see Algorithm 3.1 in Nocedal and Wright). Start with an initial step length $\bar{\alpha} = 1$ and use the parameters $c = 1/8$ (sufficient decrease parameter) and $\rho = 1/2$ (contraction factor).
(10 points)

Problem 2 Consider the constrained optimization problem

$$x^3 \to \min \qquad \text{subject to } x^2 + y^2 = 1. \tag{1}$$

Find (by whatever means) the solution of this problem. In addition, formulate the augmented Lagrangian for this problem and determine all parameters $\lambda \in \mathbb{R}$ and $\mu > 0$ for which the solution of (1) is a local minimizer of the augmented Lagrangian.
(15 points)

Problem 3 Consider the constrained optimization problem

$$f(x,y) := (x+1)^2 + (y-2)^2 \to \min \qquad \text{subject to } (x,y) \in \Omega,$$

where the set $\Omega \subset \mathbb{R}^2$ is given by the inequality constraints

$$xy \geq 0 \qquad \text{and} \qquad x - y(y+2) \geq 0.$$

a) Sketch the set Ω and find all points $(x,y) \in \Omega$ for which the LICQ holds.
(5 points)

b) Determine the set $\mathcal{F}(0,0)$ of linearized feasible direction at $(0,0)$ and show that the tangent cone to Ω at $(0,0)$ is different from $\mathcal{F}(0,0)$.
(10 points)

c) Use the second order optimality conditions in order show that the point $(0,0)$ is a local solution of the constrained optimization problem.
(15 points)

Problem 4 A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called *strongly convex*, if there exists $c > 0$ such that the function $x \mapsto f(x) - \frac{c}{2}\|x\|^2$ is convex.
A function $F\colon \mathbb{R}^\ell \times \mathbb{R}^m \to \mathbb{R}$ is called *strongly separately convex*, if there exists $c > 0$ such that for every $\hat{x} \in \mathbb{R}^\ell$ and $\hat{y} \in \mathbb{R}^m$ the functions

$$x \mapsto F(x, \hat{y}) - \frac{c}{2}\|x\|^2 \qquad \text{and} \qquad y \mapsto F(\hat{x}, y) - \frac{c}{2}\|y\|^2$$

are convex.

a) Show that a twice continuously differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}$ is strongly convex, if and only if there exists $c > 0$ such that

$$p^T \nabla^2 f(x) p \geq c\|p\|^2$$

for every $p \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$.
(5 points)

b) Show that the function $F\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$,

$$F(x, y) = x^4 + y^4 - 4xy + x^2 + y^2$$

is non-convex, but strongly separately convex.
(5 points)

c) Assume that the function $f\colon \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable and strongly convex and assume that x^* is a minimizer of f. Show that there exists $c > 0$ such that

$$f(x) \geq f(x^*) + \frac{c}{2}\|x - x^*\|^2$$

for every $x \in \mathbb{R}^n$.
(5 points)

d) Assume that the function $F\colon \mathbb{R}^\ell \times \mathbb{R}^m \to \mathbb{R}$ is strongly separately convex, twice continuously differentiable, and coercive. Given $y_0 \in \mathbb{R}^\ell$ we define iterates $x_{k+1} \in \mathbb{R}^\ell$, $y_{k+1} \in \mathbb{R}^m$ by

$$\begin{aligned} x_{k+1} &\text{ minimizes the function } x \mapsto F(x, y_k), \\ y_{k+1} &\text{ minimizes the function } y \mapsto F(x_{k+1}, y). \end{aligned}$$

Show that

$$\sum_{k \in \mathbb{N}} \|x_{k+1} - x_k\|^2 + \|y_{k+1} - y_k\|^2 < \infty.$$

(10 points)

e) Show that every strongly convex and differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}$ has a unique minimizer $x^* \in \mathbb{R}^n$.
(10 points)