Norwegian University of Science
and Technology
Department of Mathematics

TMA4180 Optimization
Theory
Spring 2015

Semester Project

1 Instructions

- The deadline for submitting the project is Friday, April 17, at 16:00.
- You may work on the project either alone or in pairs. You are free to discuss the project in larger groups, but when you write the report or do the implementations or numerical tests, you should do so in the small groups.
- Write you candidate number(s) on the top of the report; do not use names.
- Submit your reports (as PDF) and code (ZIP) by e-mail to Torbjørn. (Handwritten reports can alsoalternatively be put in my mailbox in the 7th floor, SBII.)
- The reports should be no longer than 10 pages; 5–6 pages (in LaTeXor equivalent) are perfect.
- You can write your project in English or Norwegian. Handwritten reports are fine, but they should be legible, even more so if Norwegian.
- When you refer to books or other sources of information, always provide citations.
- In your report, provide concise answers to the different questions. Additionally, provide sufficient details about the implementation (algorithm used, linesearch method and parameters, starting values, parameter updates,...). You may include code snippets of important parts of the algorithms in the report, but do not provide listings of the full code. Also, report on the algorithmic performance and try to relate the actual performance with what is predicted by theory.
- You are more than welcome to experiment with different algorithms and also more complicated problem settings. Correct and relevant extra work may—to some degree—compensate for deficiencies in other tasks.
- The project counts for 20% of the final grade. If you don't hand it in, you may still take the exam, but you cannot achieve more than 80% in the course (which corresponds to a B, but requires an almost flawless exam).

2 Hanging chain

The goal of this project is to find the static equilibrium position of a hanging chain.

We assume that the chain consists of $n+1$ rigid links of lengths $\ell_i > 0$, which are connected by perfectly flexible joints at positions $G_i = (x_i, y_i)$, $i = 0, \ldots, n+1$. The end joints of the chain are fixed at positions

$$G_0 = (x_0, y_0) := (0,0) \qquad \text{and} \qquad G_{n+1} = (x_{n+1}, y_{n+1}) := (a,b)$$

for some $(a,b) \in \mathbb{R}^2$. The only force affecting the chain is gravity.

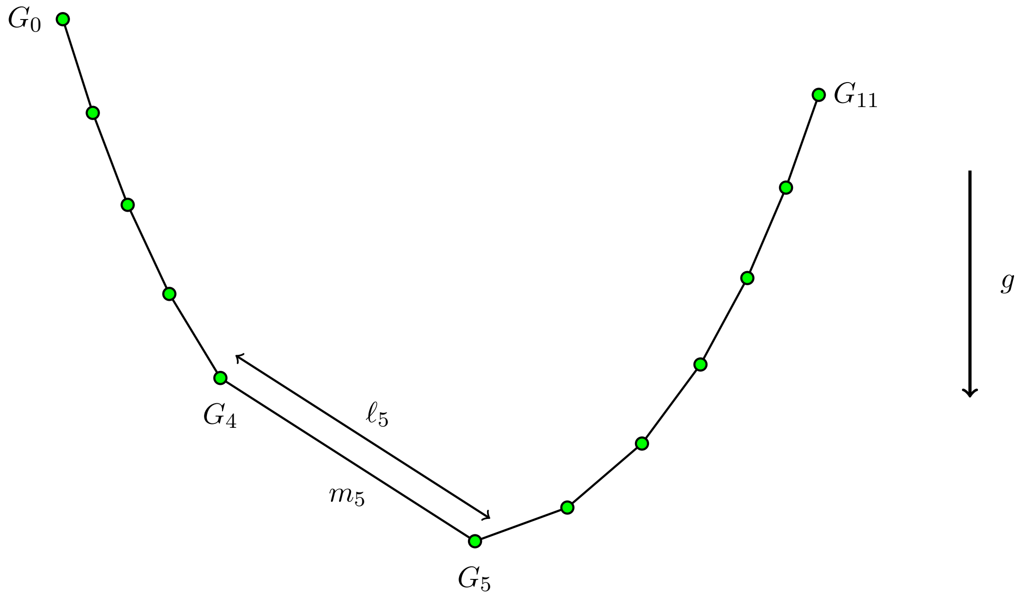

Figure 1: Hanging chain: The chain has eleven links, all the links have length 1.3 apart from the fifth one, which has length 4. The mass of each link is proportional to its length. The right endpoint of the chain is fixed at the point $G_{11} = (10, -1)$.

In such a situation, the chain will eventually admit a position where its potential energy is minimal. Assuming that the links have masses $m_i > 0$ and are homogeneous, the potential energy of the whole chain is given by

$$E(x_1, \ldots, x_n, y_1, \ldots, y_n) := \sum_{i=1}^{n+1} m_i g \frac{y_i + y_{i-1}}{2},$$

where $g \approx 9.81\text{m/s}^2$ is the gravitational acceleration at the earth's surface.

Thus the resting position of the chain solves the constrained optimization problem

$$E(x_1, \ldots, x_n, y_1, \ldots, y_n) \to \min \tag{1}$$

subject to the constraints

$$c_i(x_1, \ldots, x_n, y_1, \ldots, y_n) := (x_i - x_{i-1})^2 + (y_i - y_{i-1})^2 - \ell_i^2 = 0 \tag{2}$$

for $i = 1, \ldots, n+1$. (Note that the parameters $x_0 = 0$, $y_0 = 0$, $x_{n+1} = a$, $y_{n+1} = b$ appear in both the energy and the constraints, but are not minimized over.)

We denote in the following by Ω the set of feasible configurations of the chain. It is possible to show that Ω is non-empty if and only if each of the numbers $\|G_0 - G_{n+1}\|$, $\ell_1, \ldots, \ell_{n+1}$ is not larger than the sum of the other $n+1$ numbers. We will always assume that this condition is satisfied.

3 Theoretical analysis

The first set of tasks is concerned with some theoretical properties of the constrained optimization problem we are dealing with. The first task is concerned with the existence of solutions:

Task 1 *Show that the optimization problem given by* (1) *and* (2) *always admits a solution provided* Ω *is non-empty.*

Next we will disuss the first order necessary conditions for a local minimum of E given the constraints:

Task 2 *Formulate the KKT conditions for the constrained optimization problem given by* (1) *and* (2).

From the lecture (or Nocedal & Wright, Theorem 12.1) we know that the KKT conditions are satisfied at every local minimum, provided the *linear independence constraint qualification* (LICQ) holds at this point. In the next task, you are asked to show that, actually, the LICQ holds for almost every configuration:

Task 3 *Assume that the LICQ does not hold for a given configuration of the chain. Show that in this case all the links of the chain are parallel.*

Note that this result in particular implies that (for $G_0 \neq G_{n+1}$) there exist at most finitely many feasible configurations at which the LICQ does not hold. In addition, it is possible to show that configurations for which all links are parallel are—with two exceptions—no local minima.

The first exception is the trivial one, where the sum of the lengths of the links is precisely the distance between G_0 and G_{n+1} (i.e., $\sum_i \ell_i = \|G_0 - G_{n+1}\|$). In such a situation there is only one feasible configuration—a straight, taut chain—, which is therefore necessarily the global minimizer of the energy.

The other exception can occur if the end joints G_0 and G_{n+1} lie in the same vertical line. Then it is possible that also the minimizing configuration of the chain lies in the same vertical line. However, one can show that the KKT conditions are nevertheless satisfied at such a minimum, even though the LICQ fails:

Task 4 *Assume that* $a = 0$ *and that there exists a feasible configuration of the chain for which* $x_i = 0$ *for all* i *(that is, the whole chain is contained in a vertical line). Show that in such a situation the KKT conditions are satisfied for certain Lagrange parameters. Is such a configuration necessarily a local minimum of* E*?*

4 Augmented Lagrangian Method

Now we consider the numerical solution of this constrained optimization problem using the *Augmented Lagrangian Method* (ALM), see Nocedal & Wright, Chapter 17.3.

Task 5 *Define the augmented Lagrangian of the constrained optimization problem given by* (1) *and* (2)*. In addition, show that the augmented Lagrangian attains a global minimum for all possible Lagrange parameters* λ *and all* $\mu > 0$*.*

Task 6 *Implement an augmented Lagrangian based numerical method for the solution of the constrained optimization problem given by* (1) *and* (2)*. Use any method you like for the solution of the unconstrained optimization problem that has to be performed in each step of the ALM, but provide some explanation for your choice.*

Task 7 *Test your method at least in the following settings and discuss for the different examples the convergence behavior of both the primal variables and the Lagrange multipliers:*

1. $n = 4$, $G_5 = (1, -0.3)$*, the lengths of the links are (in order)* 0.4, 0.3, 0.25, 0.2, 0.4*, the masses are identical to the lengths.*
2. $n = 4$, $G_5 = (0, -1)$*, the lengths of the links are (in order)* 0.5, 0.5, 2.0, 0.4, 0.4*, the masses are identical to the lengths.*
3. $n = 1$, $G_2 = (2, 0)$*, the lengths of the links and their weights are all* 1*.*
4. $n = 1$, $G_2 = (0, -2)$*, the lengths of the links and their weights are all* 1*.*

Feel free to test you method also on other, more complicated examples of your choice.

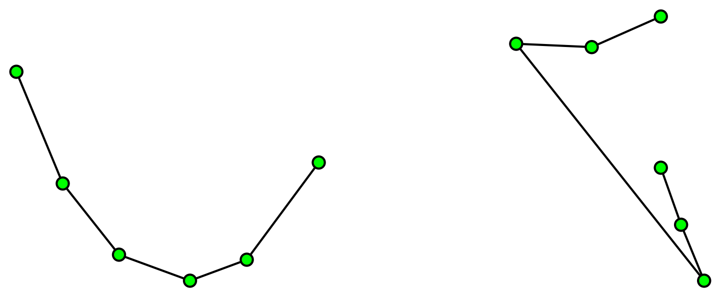

Figure 2: *Left:* Solution of Example 1 in Task 7. *Right:* One possible solution of Example 2 in Task 7.