Norwegian University of Science and Technology
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TMA4180
Optimization Theory
Spring 2015

Exercise set 8

Methods for constrained optimization

The first four exercises are concerned with different computational methods for constrained optimization and, specifically, with penalty methods, which are discussed in Nocedal & Wright, Chapter 17. Exercises 1 and 4 are purely computational (though 1 can be a bit tedious), whereas 2 and 3 are more of theoretical interest. Exercise 2 in particular shows some connection between topics in optimization and linear algebra: The matrix $A^T(AA^T)^{-1}$ that is defined there actually coincides (in this particular case) with the *Moore–Penrose pseudoinverse* $A^\dagger$ of A (cf. the lecture notes on *Linear methods*, last page, from the last term). In addition, the matrix in 2b is an approximation of the Moore–Penrose pseudoinverse; this allows another definition of $A^\dagger$ as $A^\dagger = \lim_{\mu\to\infty} A^T(\frac{1}{\mu}\operatorname{Id} + AA^T)^{-1}$, which avoids the usage of a singular value decomposition. Finally, 2c shows that the quadratic penalty method can be regarded as a relaxation of constrained optimization: instead of solving the constrained problem exactly, one allows instead for a small error in the constraints.

1 Consider the constrained optimization problem

$$\frac{1}{2}(x^2 + y^2) \to \min \qquad \text{subject to } xy = 1.$$

a) Find (by whatever means) the solutions of this problem. In addition, find the values of the corresponding Lagrange multipliers.

b) Formulate the unconstrained optimization problem that results from the application of the quadratic penalty method with parameter $\mu > 0$. Solve these problems for all possible parameters μ and verify that the solutions converge to the solutions of the constrained optimization problem as $\mu \to \infty$.

c) Formulate the augmented Lagrangian for this constrained optimization problem and find (for all possible parameters $\lambda \in \mathbb{R}$ and $\mu > 0$) the global solutions of this (unconstrained) optimization problem. For which parameters does one recover the solution of the original constrained problem?

2 Assume that $A \in \mathbb{R}^{m\times n}$ with $m < n$ is a matrix of full rank and that $b \in \mathbb{R}^m \setminus \{0\}$. Consider the optimization problem

$$\frac{1}{2}\|x\|^2 \to \min \qquad \text{subject to} \qquad Ax = b.$$

a) Formulate the KKT-conditions for this problem and show that the unique solution is given by

$$x^* = A^T(AA^T)^{-1}b.$$

b) Formulate the quadratic penalty method for this constrained optimization problem, and show that the unique minimizer with parameter $\mu > 0$ is given by

$$x_\mu := A^T\Big(\frac{1}{\mu}\,\mathrm{Id} + AA^T\Big)^{-1} b$$

with $\mathrm{Id} \in \mathbb{R}^{m\times m}$ denoting the identity matrix.

c) Now consider the optimization problem

$$\frac{1}{2}\|x\|^2 \to \min \qquad \text{subject to} \qquad \frac{1}{2}\|Ax - b\|^2 \le \varepsilon$$

for some $\varepsilon > 0$, and denote its solution by $\hat{x}_\varepsilon$. Show that either $\frac{1}{2}\|b\|^2 \le \varepsilon$ (in which case $\hat{x}_\varepsilon = 0$), or there exists $\mu > 0$ such that $\hat{x}_\varepsilon = x_\mu$.

3 Find an equality constrained optimization problem, for which the augmented Lagrangian is unbounded for all Lagrange parameters λ and every parameter $\mu > 0$.

4 Consider the optimization problem

$$\frac{1}{2}(x^2 + y^2) \to \min \qquad \text{subject to } x = 1.$$

Compute for every parameter $\mu > 0$ the minimizers of the non-smooth penalty function $\Phi_1(x;\mu)$. For which parameters do these minimizers coincide with the minimizer of the constrained problem?

Convex analysis

The following six examples are concerned with convex functions, their subdifferentials and their conjugates. You find all the theory and notation needed to solve these problems in the lecture notes on convex analysis on the course webpage.

5 Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is convex and that $g\colon \mathbb{R} \to \mathbb{R}$ is convex and monotoneously increasing. Show that the function $g \circ f$ is convex.

6 Prove that the function $f\colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, $f(x) = -\sqrt{x}$ for $x \ge 0$ and $f(x) = +\infty$, is convex, and compute its subdifferential $\partial f(x)$ for every $x \ge 0$.

7 Compute the convex conjugate of the function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$.

8 Prove that the function $f\colon \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$,

$$f(x,y) = \begin{cases} x^2/y & \text{if } y > 0, \\ 0 & \text{if } x = 0, \\ +\infty & \text{else,} \end{cases}$$

is convex and compute its subdifferential whenever it exists.

9 The *Huber function* $f\colon \mathbb{R} \to \mathbb{R}$ is defined as

$$f(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } |x| \le 1, \\ |x| - \frac{1}{2} & \text{if } |x| > 1. \end{cases}$$

Show that this function is convex, and compute its conjugate f^*.

10 Assume that $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is strictly convex and that $x \neq y \in \mathbb{R}^n$. Prove that $\partial f(x) \cap \partial f(y) = \emptyset$.

Calculus of variations

The last set of exercises is concerned with calculus of variations in a one-dimensional settings. Mainly, you are asked to derive and solve the Euler–Lagrange equations for certain variational problems. Note that it will be necessary in problem 12b to find additionally the correct boundary condition at $x = 1$.

11 Let $\ell > 0$, a, $b > 0$, and let $y\colon (0,\ell) \to \mathbb{R}_{\ge 0}$ be a non-negative function. Denote by $S(y)$ the surface of the body defined by rotating y around the x-axis. Then

$$S(y) = 2\pi \int_0^\ell y(x)\sqrt{1 + y'(x)^2}\,dx.$$

a) Formulate the Euler–Lagrange equation for this variational functional.

b) Verify that solutions of the Euler–Lagrange equation have the form

$$y(x) = A\cosh\Big(\frac{x - B}{A}\Big)$$

for parameters $A > 0$ and $B \in \mathbb{R}$.

12 Given a function $y\colon (0,1) \to \mathbb{R}$ define

$$F(y) = \frac{1}{2}\int_0^1 y(x)^2 + y'(x)^2 \, dx.$$

a) Minimize F subject to the constraints $y(0) = 1$ and $y(1) = 0$.

b) Minimize F subject to the constraint $y(0) = 1$.

13 Minimize the functional

$$F(y) = \int_0^2 e^{-x} y'(x)^2 - y(x) \, dx$$

subject to the constraints $y(0) = 1$ and $y(2) = 0$.