Norwegian University of Science and Technology
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TMA4180 Optimization Theory Spring 2015

Exercise set 6 solution

1 **a)** We wish to reformulate the unconstrained optimization problem

$$\min_{(a,b)\in\mathbb{R}^2} f(a,b), \quad f(a,b) = \sum_{i=1}^{m} |at - i + b - s_i|. \tag{1}$$

The problem can be stated as a linear programme by introducing artificial variables u_i where $i = 1...m$, writing $\mathbf{x} = [u_1, ...u_m, a, b]^T$ and considering the linear optimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^{m+2}} \sum_{i=1}^{m} u_i, \quad \text{s.t.} \quad \begin{cases} u_i \geq at_i + b - s_i \\ u_i \geq -(at_i + b - s_i) \end{cases} \quad i = 1, ..., m. \tag{2}$$

This is an equivalent problem to (1) since an optimal solution will always be such that $u_i = |at_i + b - s_i|$. The argument for this is as following: Pick some $j \in 1, ..., m$. If

$$u_j > \max\{at_j + b - s_j, -(at_j + b - s_j)\} = |at_j + b - s_j|,$$

then the objective function value can be reduced further by lowering u_j such that $u_j = |at_j + b - s_j|$. Therefore, any optimal solution to (2) will have $u_i = |at_i + b - s_i|$.

b) To formulate the KKT conditions for problem (2), we first state it in the standard form (still with $\mathbf{x} = [u_1, ...u_m, a, b]^T$):

$$\min_{\mathbf{x}\in\mathbb{R}^{m+2}} \sum_{i=1}^{m} u_i, \quad \text{s.t.} \quad \begin{cases} u_i - at_i - b + s_i \geq 0 \\ u_i + at_i + b - s_i \geq 0 \end{cases} \quad i = 1, ..., m.$$

and identify the $2m$ inequality constraint functions as following:

$$\begin{aligned} c_i(\mathbf{x}) &= u_i - at_i - b + s_i, \quad i = 1, ..., m \\ c_i(\mathbf{x}) &= u_{i-m} + at_{i-m} + b - s_{i-m}, \quad i = m+1, ..., 2m. \end{aligned}$$

The Lagrangian function becomes

$$\begin{aligned} \mathcal{L}(\mathbf{x}, \lambda) &= \sum_{i=1}^{m} u_i - \sum_{i=1}^{2m} \lambda_i c_i(\mathbf{x}) \\ &= \sum_{i=1}^{m} u_i - \sum_{i=1}^{m} \lambda_i (u_i - at_i - b + s_i) - \sum_{i=1}^{m} \lambda_{i+m}(u_i + at_i + b - s_i) \end{aligned}$$

with gradient

$$\nabla_x \mathcal{L}(\mathbf{x}, \lambda) = \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial u_1} \\ \vdots \\ \frac{\partial \mathcal{L}}{\partial u_m} \\ \frac{\partial \mathcal{L}}{\partial a} \\ \frac{\partial \mathcal{L}}{\partial b} \end{bmatrix} = \begin{bmatrix} 1 - \lambda_1 - \lambda_{1+m} \\ \vdots \\ 1 - \lambda_m - \lambda_{2m} \\ \sum_{i=1}^m \lambda_i t_i - \sum_{i=1}^m \lambda_{i+m} t_i \\ \sum_{i=1}^m \lambda_i - \sum_{i=1}^m \lambda_{i+m} \end{bmatrix}$$

Using the above definitions of $\nabla_x \mathcal{L}(\mathbf{x}, \lambda)$ and $c_i(\mathbf{x})$, we can state the KKT conditions as following:

$$\begin{aligned} \nabla_x \mathcal{L}(\mathbf{x}^*, \lambda^*) &= 0, \\ c_i(\mathbf{x}^*) &\geq 0, \quad i = 1, ...2m \\ \lambda_i^* &\geq 0, \quad i = 1, ...2m \\ \lambda_i^* c_i(\mathbf{x}^*) &= 0, \quad i = 1, ...2m. \end{aligned}$$

$\boxed{2}$ **a)** We begin by stating the problem in standard form, writing $\mathbf{x} = [x, y]^T$:

$$\min_{\mathbf{x} \in \mathbb{R}^2} f(\mathbf{x}) \quad \text{s.t.} \quad c_i(\mathbf{x}) \geq 0, \quad i = 1, 2, 3,$$

where

$$\begin{aligned} f(\mathbf{x}) &= x^2 + y^2, \\ c_1(\mathbf{x}) &= x + y - 1, \\ c_2(\mathbf{x}) &= 2 - y, \\ c_3(\mathbf{x}) &= y^2 - x. \end{aligned}$$

We then find the Lagrangian function and its gradient:

$$\begin{aligned} \mathcal{L}(\mathbf{x}, \lambda) &= x^2 + y^2 - \lambda_1(x + y - 1) - \lambda_2(2 - y) - \lambda_3(y^2 - x) \\ \nabla_x \mathcal{L}(\mathbf{x}, \lambda) &= \begin{bmatrix} 2x - \lambda_1 + \lambda_3 \\ 2y - \lambda_1 + \lambda_2 - 2y\lambda_3 \end{bmatrix}. \end{aligned}$$

The KKT conditions can now be stated in full as:

$$2x^* - \lambda_1^* + \lambda_3^* = 0 \tag{3a}$$
$$2y^* - \lambda_1^* + \lambda_2^* - 2y^*\lambda_3^* = 0 \tag{3b}$$
$$x^* + y^* - 1 \geq 0 \tag{3c}$$
$$2 - y^* \geq 0 \tag{3d}$$
$$y^{*2} - x^* \geq 0 \tag{3e}$$
$$\lambda_i^* \geq 0, \quad i = 1, 2, 3 \tag{3f}$$
$$\lambda_1^*(x^* + y^* - 1) = 0 \tag{3g}$$
$$\lambda_2^*(2 - y^*) = 0 \tag{3h}$$
$$\lambda_3^*(y^{*2} - x^*) = 0. \tag{3i}$$

b) The feasible set is sketched in figure 1. We will find all KKT points by systematically considering all possible active sets of constraints. Remember that a constraint c_i is active at a point $\mathbf{x}$ if $c_i(\mathbf{x}) = 0$, and that if all λ_i^* are negative at a point, then it is a candidate for a maximizer. Also, the LICQ conditions are satisfied at every point we consider here; with one active constraint, the LICQ conditions hold trivially, and in the cases with two constraints it is not hard to check that the LICQ conditions do hold.

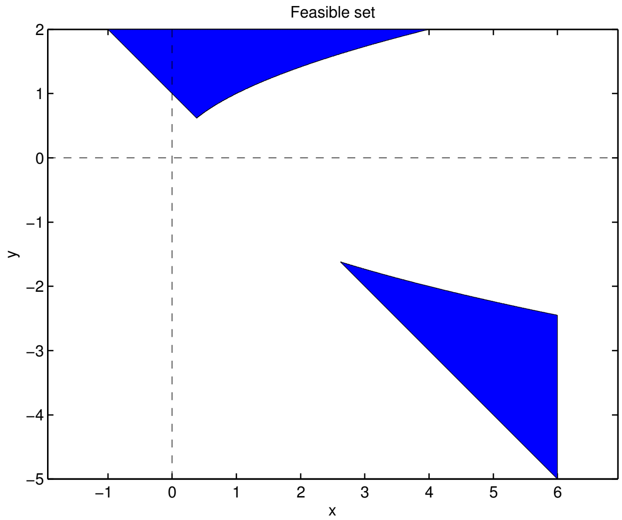

Figure 1: Feasible set. Note: The lower "triangle" extends further toward infinity.

Observe that if $\mathbf{x}^* = [x^*, y^*]^T$ is a KKT point, then from (3a) and (3b) we have:

$$x^* = \frac{\lambda_1^* - \lambda_3^*}{2}, \quad y^* = \frac{\lambda_1^* - \lambda_2^*}{2(1 - \lambda_3^*)}.$$

From here on, we will drop the asterisk in the notation and write x for x^*, etc.

First, if the active set is empty, i.e. neither of (3c)-(3e) are equalities. This corresponds to the interior of the domain. Then, by (3g)-(3i), we have $\lambda_1 = \lambda_2 = \lambda_3 = 0$, and so $x = y = 0$. But this point is not feasible, since it violates condition (3c). Thus, with the active set empty, there are no KKT points.

Next, we consider the case when the active set contains one index, i.e. exactly one of (3c)-(3e) is an equality. This corresponds to the boundaries of the domain, excepting the corner points. If (3c) is active, then $\lambda_2 = \lambda_3 = 0$ while $\lambda_1 \geq 0$. We get

$$x = \frac{\lambda_1}{2}, \quad y = \frac{\lambda_1}{2},$$

and inserting this into (3c) (which is now an equality), we get the condition

$$\frac{\lambda_1}{2} + \frac{\lambda_1}{2} - 1 = 0 \Rightarrow \lambda_1 = 1,$$

giving us the point $(x, y) = (\frac{1}{2}, \frac{1}{2})$. But this point violates condition (3e), so $(\frac{1}{2}, \frac{1}{2})$ is not a KKT point.

If (3d) is active, then $\lambda_1 = \lambda_3 = 0$ while $\lambda_2 \geq 0$, so

$$x = 0, \quad y = -\frac{\lambda_2}{2}.$$

Inserting this into the equality (3d), we get

$$2 + \frac{\lambda_2}{2} = 0 \Rightarrow \lambda_2 = -4.$$

Thus, (0,2) is a candidate for a maximizer. One can then check to verify that all KKT conditions are satisfied, and we find (0,2) to be a KKT point corresponding to a maximizer.

If (3e) is active, then $\lambda_1 = \lambda_2 = 0$ while $\lambda_3 \geq 0$, so

$$x = -\frac{\lambda_3}{2}, \quad y = 0.$$

Inserting this into the equality (3e), we get

$$\frac{\lambda_3}{2} = 0 \Rightarrow \lambda_3 = 0.$$

This gives the candidate point $(0, 0)$, which is not feasible since it violates (3c), and thereby is not a KKT point.

Having considered all possible active sets of one index, we now turn to the cases with two indices, i.e. exactly two of (3c)-(3e) are equalities. This corresponds to the corner points of the domain. First, if (3c) and (3d) are both active, then $\lambda_3 = 0$ while $\lambda_1, \lambda_2 \geq 0$. This gives us

$$x = \frac{\lambda_1}{2}, \quad y = \frac{\lambda_1 - \lambda_2}{2}.$$

Plugging this into equalities (3c) and (3d) yields:

$$\begin{aligned} \frac{\lambda_1}{2} + \frac{\lambda_1 - \lambda_2}{2} - 1 = 0 \\ 2 - \frac{\lambda_1 - \lambda_2}{2} = 0, \end{aligned}$$

with solutions $\lambda_1 = -2$ and $\lambda_2 = -6$, yielding the KKT point (-1,2). Note that this is a candidate for a local maximizer, since all multipliers are negative.

Next, if (3c) and (3e) are both active, then $\lambda_2 = 0$ while $\lambda_1, \lambda_3 \geq 0$, which means

$$x = \frac{\lambda_1 - \lambda_3}{2}, \quad y = \frac{\lambda_1}{2(1 - \lambda_3)}.$$

Plugging this into equalities (3c) and (3e) yields:

$$\begin{aligned} \frac{\lambda_1 - \lambda_3}{2} + \frac{\lambda_1}{2(1 - \lambda_3)} - 1 = 0 \\ \frac{\lambda_1^2}{4(1 - \lambda_3)^2} - \frac{\lambda_1 - \lambda_3}{2} = 0. \end{aligned}$$

Solving this set of equations yields $\lambda_1 = 5 \pm \frac{9}{\sqrt{5}}$ and $\lambda_3 = 2 \pm \frac{4}{\sqrt{5}}$, thereby giving the candidate points $(x,y) = (\frac{1}{2}(3 \pm \sqrt{5}), \frac{1}{2}(-1 \mp \sqrt{5}))$ which both satisfy the KKT conditions. Since $\lambda_1, \lambda_3 \geq 0$, these points are minimizer candidates. Note: This result can be arrived upon by the easier approach of first finding the points (x,y) where c_1 and c_3 are both active, then working out what λ_1 and λ_3 are.

Finally, we check the case where (3d) and (3e) are both active, i.e. $\lambda_1 = 0$ while $\lambda_2, \lambda_3 \geq 0$. This gives us

$$x = -\frac{\lambda_3}{2}, \quad y = -\frac{\lambda_2}{2(1-\lambda_3)}.$$

Plugging this into equalities (3d) and (3e) yields:

$$\begin{aligned} 2 + \frac{\lambda_2}{2(1-\lambda_3)} &= 0 \\ \frac{\lambda_2^2}{4(1-\lambda_3)^2} + \frac{\lambda_3}{2} &= 0, \end{aligned}$$

which can be solved to find $\lambda_2 = -28$ and $\lambda_3 = -8$, giving the candidate point $(x,y) = (4,2)$, which is a candidate for a maximizer, since the multipliers are negative.

Concerning the case with all constraints active, we may conclude that no KKT point exists; all three constraint functions cannot be active at the same point. The KKT points and their corresponding multipliers are summarized in the table below.

Point	λ_1	λ_2	λ_3	Minimizer/maximizer candidate
(0,2)	0	-4	0	Maximizer
$(\frac{1}{2}(3+\sqrt{5}), \frac{1}{2}(-1-\sqrt{5}))$	$5+\frac{9}{\sqrt{5}}$	0	$2+\frac{4}{\sqrt{5}}$	Minimizer
$(\frac{1}{2}(3-\sqrt{5}), \frac{1}{2}(-1+\sqrt{5}))$	$5-\frac{9}{\sqrt{5}}$	0	$2-\frac{4}{\sqrt{5}}$	Minimizer
(-1,2)	-2	-6	0	Maximizer
(4,2)	0	-28	-8	Maximizer

c) To determine whether the KKT points are in fact local maximizers or minimizers, we first check the second order sufficient conditions from Theorem 12.6 in N&W, i.e. whether

$$w^T \nabla^2_{xx} \mathcal{L}(x,\lambda) w > 0 \ \forall w \in \mathcal{C}(x,\lambda), w \neq 0, \tag{4}$$

where, $\mathcal{C}(x,\lambda)$ is the critical cone at x, given by (12.53) in N&W. Note that for a maximizer, we require

$$w^T \nabla^2_{xx} \mathcal{L}(x,\lambda) w < 0 \ \forall w \in \mathcal{C}(x,\lambda), w \neq 0, \tag{5}$$

and any inequality in the definition of the critical cone will be in the opposite direction.

For the four points we found with two active constraints, i.e. (-1,2), (4,2) and $(\frac{1}{2}(3 \pm \sqrt{5}), \frac{1}{2}(-1 \mp \sqrt{5}))$, we have that the critical cone is simply given as $\mathcal{C}(x,\lambda) = \{0\}$. This is because any $w \in \mathcal{C}(x,\lambda)$ must be orthogonal to the

$\nabla c_i(x)$ for which $\lambda_i > 0$ (resp. $\lambda_i < 0$ for maximizers), of which there are two for each point. Since the LICQ conditions hold at all the points considered, these two vectors are linearily independent and thus span $\mathbb{R}^2$. The only vector orthogonal to $\mathbb{R}^2$ is the zero vector. Thereby, the only vector in $\mathcal{C}(x, \lambda)$ is the zero vector for these points, and thus condition (4) (resp. condition (5) for maximizers) hold by default. We can conclude that $(\frac{1}{2}(3 \pm \sqrt{5}), \frac{1}{2}(-1 \mp \sqrt{5}))$ are strict local minimizers, and that (-1,2) and (4,2) are strict local maximizers.

Concerning the maximizer candidate (0,2), we can see that it does not satisfy (5), as we shall see now. The critical cone at this point is $\{w : w = [w_1, 0]^T\}$, since $\nabla c_2(x) = [0, -1]^T$. However, the Hessian of the Lagrange function is

$$\nabla^2_{xx}\mathcal{L}(x, \lambda) = \begin{bmatrix} 2 & 0 \\ 0 & 2(1-\lambda_3) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

and so, clearly

$$w^T \nabla^2_{xx}\mathcal{L}(x, \lambda) w > 0,$$

meaning (5) is not satisfied. This gives us an indication that (0,2) is not a maximizer, which indeed it is not. If we look at the function values $\phi(x) = f(x, 2) = 4 + x^2$, we see that f is increasing in the x direction at the point (0,2), so (0,2) cannot be a local maximizer.

In closing , we note that $f(\frac{1}{2}(3-\sqrt{5}), \frac{1}{2}(-1+\sqrt{5})) < f(\frac{1}{2}(3+\sqrt{5}), \frac{1}{2}(-1-\sqrt{5}))$, so $(\frac{1}{2}(3-\sqrt{5}), \frac{1}{2}(-1+\sqrt{5}))$ is a global minimizer and $(\frac{1}{2}(3+\sqrt{5}), \frac{1}{2}(-1-\sqrt{5}))$ is a local minimizer, while (-1,2) and (4,2) are both local maximizers since a global maximizer does not exist since $f(\mathbf{x}) \to \infty$ in the unbounded region of the feasible domain.

3 **a)** We begin by stating the problem in standard form, writing $\mathbf{x} = [x, y]^T$:

$$\min_{\mathbf{x} \in \mathbb{R}^2} f(\mathbf{x}) \quad \text{s.t.} \quad c_i(\mathbf{x}) \geq 0, \quad i = 1, 2$$

where

$$\begin{aligned} f(\mathbf{x}) &= xy, \\ c_1(\mathbf{x}) &= y - x \\ c_2(\mathbf{x}) &= x^3 - y^4 \end{aligned}$$

The feasible set is sketched in figure 2. To characterize the tangent cone $T(\mathbf{x})$ and the set of linearized feasible directions $\mathcal{F}(\mathbf{x})$, we employ lemma 12.2 in N&W, which states that if the LICQ conditions hold at a feasible point $\mathbf{x}$, then $T(\mathbf{x}) = \mathcal{F}(\mathbf{x})$. In the interior of Ω, there are no active constraints, and so the LICQ condition is vacuously true. Also, since no constraints are active, $T(\mathbf{x}) = \mathcal{F}(\mathbf{x}) = \mathbb{R}^2$.

Next, we look at the points with one active constraint, starting with the line $c_1(\mathbf{x}) = 0$ (excluding the points where $c_2(\mathbf{x}) = 0$), and observing

$$\nabla c_1(\mathbf{x}) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

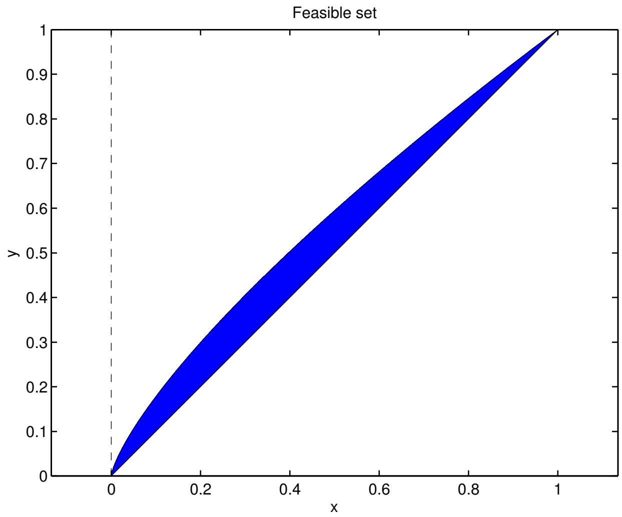

Figure 2: Feasible set, exercise 3.

This is nonzero, and so the LICQ condition holds. We find that

$$T(\mathbf{x}) = \mathcal{F}(\mathbf{x}) = \{d \in \mathbb{R}^2 : \nabla c_1(\mathbf{x})^T d \geq 0\} = \{d \in \mathbb{R}^2 : d_2 \geq d_1\}$$

Considering the line $c_2(\mathbf{x}) = 0$ (excluding the points where $c_1(\mathbf{x}) = 0$), we observe

$$\nabla c_2(\mathbf{x}) = \begin{bmatrix} 3x^2 \\ -4y^3 \end{bmatrix},$$

which is also nonzero as the point (0,0) is excluded. Thereby, the LICQ conditions hold, and we have

$$T(\mathbf{x}) = \mathcal{F}(\mathbf{x}) = \{d \in \mathbb{R}^2 : \nabla c_2(\mathbf{x})^T d \geq 0\} = \{d \in \mathbb{R}^2 : 3x^2 d_1 \geq 4y^3 d_2\}.$$

Lastly, we consider the corner points (1,1) and (0,0), where both constraints are active. In (1,1) we have

$$\nabla c_1(1,1) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \nabla c_2(1,1) = \begin{bmatrix} 3 \\ -4 \end{bmatrix},$$

so the LICQ condition holds. Thereby,

$$\begin{aligned} T(1,1) = \mathcal{F}(1,1) &= \{d \in \mathbb{R}^2 : \nabla c_1(1,1)^T d \geq 0 \text{ and } \nabla c_2(1,1)^T d \geq 0\} \\ &= \{d \in \mathbb{R}^2 : d_2 \geq d_1 \text{ and } 3d_2 \geq 4d_1\}. \end{aligned}$$

In the last point, (0,0), the LICQ condition does not hold, since

$$\nabla c_1(0,0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \nabla c_2(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore, we cannot expect that $T(0,0) = \mathcal{F}(0,0)$. Indeed, we have

$$\begin{aligned} \mathcal{F}(0,0) &= \{d \in \mathbb{R}^2 : \nabla c_1(0,0)^T d \geq 0 \text{ and } \nabla c_2(0,0)^T d \geq 0\} \\ &= \{d \in \mathbb{R}^2 : d_2 \geq d_1\}, \end{aligned}$$

defining a half-space. On the other hand, we can find the tangent cone by looking at the limiting vectors along the lines $c_1(\mathbf{x}) = 0$ and $c_2(\mathbf{x}) = 0$ as $\mathbf{x} \to 0$. Traveling toward (0,0) along $c_1(\mathbf{x}) = 0$, we take

$$z_k = \begin{bmatrix} 1/k \\ 1/k \end{bmatrix}, \quad t_k = ||z_k|| = \sqrt{2}/k,$$

and find the limiting direction

$$d = \lim_{k\to\infty} \frac{z_k}{t_k} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

Along $c_2(\mathbf{x}) = 0$, we take

$$z_k = \begin{bmatrix} 1/k \\ 1/k^{3/4} \end{bmatrix}, \quad t_k = ||z_k|| = \frac{\sqrt{\sqrt{k}+1}}{k},$$

and find the limiting direction

$$d = \lim_{k\to\infty} \frac{z_k}{t_k} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The tangent cone at (0,0) contains all vectors between these limiting cases, which can be shown to be:

$$T(0,0) = \{d \in \mathbb{R}^2 : d_1 \geq 0 \text{ and } d_2 \geq d_1\}.$$

b) We find the Lagrange function and its gradient:

$$\mathcal{L}(\mathbf{x}, \lambda) = xy - \lambda_1(y - x) - \lambda_2(x^3 - y^4)$$
$$\nabla_x \mathcal{L}(\mathbf{x}, \lambda) = \begin{bmatrix} y + \lambda_1 - 3x^2\lambda_2 \\ x - \lambda_1 + 4y^3\lambda_2 \end{bmatrix},$$

and state the KKT conditions as:

$$y + \lambda_1 - 3x^2\lambda_2 = 0 \tag{6a}$$
$$x - \lambda_1 + 4y^3\lambda_2 = 0 \tag{6b}$$
$$y - x \geq 0 \tag{6c}$$
$$x^3 - y^4 \geq 0 \tag{6d}$$
$$\lambda_i \geq 0, \quad i = 1, 2 \tag{6e}$$
$$\lambda_1(y - x) = 0 \tag{6f}$$
$$\lambda_2(x^3 - y^4) = 0. \tag{6g}$$

Now, we can check the different cases of active constraints to find KKT points. First, with no active constraints, i.e. $\lambda_1 = \lambda_2 = 0$, we get the point (0,0). In fact, this is a point with both constraints active; it just so happens that $\lambda_1 = \lambda_2 = 0$ here. As seen in a), the LICQ does not hold here, but it is still a KKT point because the gradient of f at (0,0) (which is 0) can be written as a non-negative linear combination of the gradients of the constraints. Thereby, we conclude that there exist no KKT points with neither constraint active but that (0,0) is a KKT point. What the failure of the LICQ at (0,0) implies is the following: There exists a function f which has a local minimum at (0,0), but for which (0,0) is no KKT point. However, we might still be lucky for any given function f, as is the case here.

Next, we check with one active constraint. First, with $\lambda_1 \geq 0, \lambda_2 = 0$, we have from (6a) and (6b) that $x = \lambda_1$ and $y = -\lambda_1$. Inserting into the equality (6c) yields $\lambda_1 = 0$, and therefore $(x, y) = (0, 0)$ again, which is not a KKT point.

With $\lambda_2 \geq 0, \lambda_1 = 0$, equations (6a), (6b) and (6d) become

$$\begin{aligned} y - 3x^2\lambda_2 &= 0, \\ x + 4y^3\lambda_2 &= 0, \\ x^3 &= y^4. \end{aligned}$$

Multiplying the first of these by x, the second by y, applying the third and adding the two first gives

$$y^4\lambda_2 = 0.$$

Any solution of this leads to the point (0,0), which already discovered as a KKT point.

Finally, we check the case with two active constraints, for which there are two points; (0,0), which is already considered, and (1,1). In the point (1,1), we find (by (6a) and (6b)) that $\lambda_1 = -7$ and $\lambda_2 = -2$. All KKT conditions for maximizers are satisfied, so (1,1) is a KKT point candidate for a maximizer. Since the LICQ holds here, with two linearly independent ∇c_i, we know from the discussion in the previous exercise that this is a local maximizer.

The only undecided point left is (0,0), for which the LICQ did not hold, and which therefore can be neither confirmed or discarded as a minimizer/maximizer using the second order necessary and sufficient conditions. However, it is clearly a local minimum, as we are only considering nonnegative values for x and y, and since $f(x, y) = xy$, its global minimum (in the feasible set) is located at (0,0).

4 **a)** We begin by stating the problem in standard form, writing $\mathbf{x} = [x, y]^T$:

$$\min_{\mathbf{x}\in\mathbb{R}^2} f(\mathbf{x}) \quad \text{s.t.} \quad c_i(\mathbf{x}) \geq 0, \quad i = 1, 2$$

where

$$\begin{aligned} f(\mathbf{x}) &= x, \\ c_1(\mathbf{x}) &= y - x^4 \\ c_2(\mathbf{x}) &= x^3 - y \end{aligned}$$

The feasible set is sketched in figure 3. To characterize the tangent cone $T(\mathbf{x})$ and the set of linearized feasible directions $\mathcal{F}(\mathbf{x})$, we employ lemma 12.2 in N&W as in the previous exercise. In the interior of Ω, there are no active constraints, and so the LICQ condition is vacuously true. Also, since no constraints are active, $T(\mathbf{x}) = \mathcal{F}(\mathbf{x}) = \mathbb{R}^2$.

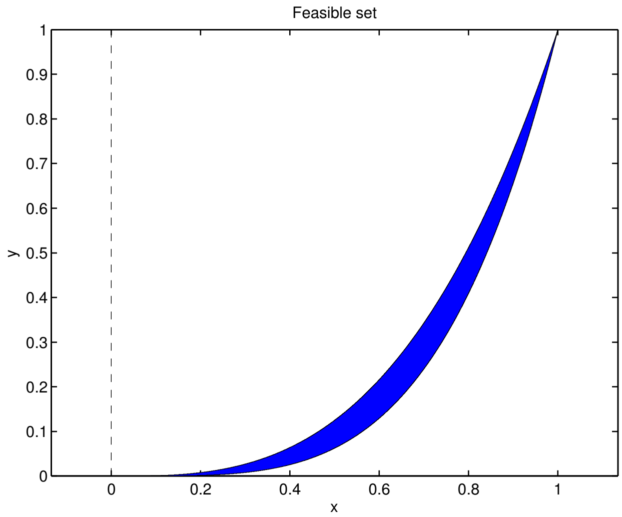

Figure 3: Feasible set.

Next, we look at the points with one active constraint, starting with the line $c_1(\mathbf{x}) = 0$ (excluding the points where $c_2(\mathbf{x}) = 0$), and observing

$$\nabla c_1(\mathbf{x}) = \begin{bmatrix} -4x^3 \\ 1 \end{bmatrix}$$

This is nonzero, and so the LICQ condition holds. We find that

$$T(\mathbf{x}) = \mathcal{F}(\mathbf{x}) = \{d \in \mathbb{R}^2 : \nabla c_1(\mathbf{x})^T d \geq 0\} = \{d \in \mathbb{R}^2 : d_2 \geq 4x^3 d_1\}$$

Considering the line $c_2(\mathbf{x}) = 0$ (excluding the points where $c_1(\mathbf{x}) = 0$), we observe

$$\nabla c_2(\mathbf{x}) = \begin{bmatrix} 3x^2 \\ -1 \end{bmatrix},$$

which is also nonzero. Thereby, the LICQ conditions hold, and we have

$$T(\mathbf{x}) = \mathcal{F}(\mathbf{x}) = \{d \in \mathbb{R}^2 : \nabla c_2(\mathbf{x})^T d \geq 0\} = \{d \in \mathbb{R}^2 : 3x^2 d_1 \geq d_2\}.$$

Lastly, we consider the corner points (1,1) and (0,0), where both constraints are active. In (1,1) we have

$$\nabla c_1(1,1) = \begin{bmatrix} -4 \\ 1 \end{bmatrix} \quad \text{and} \quad \nabla c_2(1,1) = \begin{bmatrix} 3 \\ -1 \end{bmatrix},$$

so the LICQ condition holds. Thereby,

$$\begin{aligned} T(1,1) = \mathcal{F}(1,1) &= \{d \in \mathbb{R}^2 : \nabla c_1(1,1)^T d \geq 0 \text{ and } \nabla c_2(1,1)^T d \geq 0\} \\ &= \{d \in \mathbb{R}^2 : d_2 \geq 4d_1 \text{ and } 3d_1 \geq d_2\}. \end{aligned}$$

In the last point, (0,0), the LICQ condition does not hold, since

$$\nabla c_1(0,0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \nabla c_2(0,0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Therefore, we cannot generally expect that $T(0,0) = \mathcal{F}(0,0)$. We have

$$\begin{aligned}\mathcal{F}(0,0) &= \{d \in \mathbb{R}^2 : \nabla c_1(0,0)^T d \geq 0 \text{ and } \nabla c_2(0,0)^T d \geq 0\} \\ &= \{d \in \mathbb{R}^2 : d_2 \geq 0 \text{ and } d_2 \leq 0\} \\ &= \{d \in \mathbb{R}^2 : d_2 = 0\} = \{d \in \mathbb{R}^2 : d = [d_1, 0]^T\}.\end{aligned}$$

We can find the tangent cone by looking at the limiting vectors along the lines $c_1(\mathbf{x}) = 0$ and $c_2(\mathbf{x}) = 0$ as $\mathbf{x} \to 0$. Traveling toward (0,0) along $c_1(\mathbf{x}) = 0$, we take

$$z_k = \begin{bmatrix} 1/k \\ 1/k^4 \end{bmatrix}, \quad t_k = ||z_k|| = \frac{k^4}{\sqrt{k^6+1}}$$

and find the limiting direction

$$d = \lim_{k\to\infty} \frac{z_k}{t_k} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Along $c_2(\mathbf{x}) = 0$, we take

$$z_k = \begin{bmatrix} 1/k \\ 1/k^3 \end{bmatrix}, \quad t_k = ||z_k|| = \frac{k^3}{\sqrt{k^4+1}},$$

and find the limiting direction

$$d = \lim_{k\to\infty} \frac{z_k}{t_k} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Thus, the tangent cone at (0,0) is

$$T(0,0) = \{d \in \mathbb{R}^2 : d = [d_1, 0]^T, d_1 > 0\},$$

and we see that it does not coincide with the set of linearized feasible directions.

b) We find the Lagrange function and its gradient:

$$\begin{aligned}\mathcal{L}(\mathbf{x}, \lambda) &= x - \lambda_1(y - x^4) - \lambda_2(x^3 - y) \\ \nabla_x \mathcal{L}(\mathbf{x}, \lambda) &= \begin{bmatrix} 1 + 4x^3\lambda_1 - 3x^2\lambda_2 \\ -\lambda_1 + \lambda_2 \end{bmatrix},\end{aligned}$$

and state the KKT conditions as:

$$1 + 4x^3\lambda_1 - 3x^2\lambda_2 = 0 \tag{7a}$$
$$-\lambda_1 + \lambda_2 = 0 \tag{7b}$$
$$y - x^4 \geq 0 \tag{7c}$$
$$x^3 - y \geq 0 \tag{7d}$$
$$\lambda_i \geq 0, \quad i = 1,2 \tag{7e}$$
$$\lambda_1(y - x^4) = 0 \tag{7f}$$
$$\lambda_2(x^3 - y) = 0. \tag{7g}$$

Now, we can take a shortcut; from (7b), we see that $\lambda_1 = \lambda_2$, and from (7a) we see that there cannot exist any KKT point for which $\lambda_1 = \lambda_2 = 0$. Therefore, the cases with no active constraints ($\lambda_1 = \lambda_2 = 0$) and one active constraint ($\lambda_1 = 0$ or $\lambda_2 = 0$) cannot produce KKT points. We are left with considering the case where both constraints are active, i.e. the corner points (0,0) and (1,1).

In the point (1,1), we find (by (7a) and (7b)) that $\lambda_1 = \lambda_2 = -1$. All KKT conditions for maximizers are satisfied, so (1,1) is a KKT point candidate for a maximizer. Since the LICQ holds here, with two linearly independent ∇c_i, we know from the discussion in the previous exercise that this is a local maximizer.

The last point is (0,0), for which we cannot write the gradient of f at (0,0) (which is $[1, 0]^T$) as a non-negative linear combination of the gradients of the constraints, and which therefore is not a KKT point. However, it is clearly a local minimum, as no other points with $x = 0$ are feasible.