NTNU – Trondheim
Norwegian University of
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Department of Mathematical Sciences

Examination paper for **TMA4180 Optimization Theory**

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Examination date: 06th June 2015

Examination time (from–to): 09:00–13:00

Permitted examination support material:

- The textbook: Nocedal & Wright, Numerical Optimization including errata.
- Rottmann, Mathematical formulae.
- Handouts on *Minimizers of optimization problems*, *Basics of convex analysis*, and *Basics of calculus of variations*.
- Approved basic calculator.

Other information:

- All answers should be justified and include enough details to make it clear which methods or results have been used.
- You may answer to the questions of the exam either in English or in Norwegian.

Language: English

Number of pages: 2

Number pages enclosed: 0

Checked by:

Date Signature

Problem 1 Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = x^4 - 2x^3 + 2x^2 - 2xy + y^2.$$

a) Compute all stationary points of f and find all local or global minimizers of f.

b) Determine whether the function f is convex or not.

c) Starting at the point $(x,y) = (0,1)$ compute one step of the steepest descent method with backtracking (Armijo) linesearch (see Algorithm 3.1 in Nocedal and Wright). Start with an initial step length $\bar{\alpha} = 1$ and use the parameters $c = 0.25$ (sufficient decrease parameter) and $\rho = 0.1$ (contraction factor).

Problem 2 Find all parameters $\alpha \in \mathbb{R}$ for which the point $(x,y) = (3,1)$ is a local solution of the optimization problem

$$x + \alpha y \to \min$$

subject to the constraints

$$\begin{aligned} xy - 3 &\geq 0, \\ 10 - x^2 - y^2 &\geq 0. \end{aligned}$$

Problem 3 Consider the optimization problem

$$x^2 + 2xy + 2y^2 \to \min \qquad \text{subject to} \qquad x + y - 1 = 0. \tag{1}$$

The unique (local and global) solution of this problem is the point $(x,y) = (1,0)$ (you don't have to show this).

a) Formulate the unconstrained optimization problem that results from the application of the quadratic penalty method to (1), and compute the solution for all possible penalty parameters $\mu > 0$.

b) Formulate the augmented Lagrangian $\mathcal{L}_A(x,\lambda;\mu)$ corresponding to (1) and compute its minimizers for all possible parameters $\lambda \in \mathbb{R}$ and $\mu > 0$. For which parameters does the minimizer of the augmented Lagrangian coincide with the solution of (1)?

Problem 4 The set $\Omega \subset \mathbb{R}^2$ is given by the constraints

$$\begin{aligned} x + 1 &\geq 0, \\ 1 - x - y &\geq 0, \\ (x+1)^2 y^3 &\geq 0. \end{aligned}$$

Using a set of suitable linear inequalities and equalities, describe both the tangent cone and the cone of linearized feasible directions for Ω at $(x, y) = (1, 0)$.

Problem 5 Show that a (not necessarily differentiable) function $f\colon \mathbb{R}^n \to \mathbb{R}_{>0}$ is convex, if the function $x \mapsto \ln(f(x))$ is convex.

Problem 6 Assume that the sequence $(x_k)_{k\in\mathbb{N}}$ is generated by the gradient descent method with backtracking linesearch for the minimization of a function f, and that $\nabla f(x_k) \neq 0$ for all k. Assume moreover that $\bar{x}$ is an accumulation point of the sequence $(x_k)_{k\in\mathbb{N}}$. Show that $\bar{x}$ is not a local maximum of f.

Problem 7 We consider a line search method of the form $x_{k+1} = x_k + \alpha_k p_k$ for the minimization of the function $f\colon \mathbb{R}^n \to \mathbb{R}$, where the search direction p_k is given as

$$p_k = -\operatorname{sgn}((\nabla f(x_k))_i)\, e_i,$$

where the index i is chosen such that $|(\nabla f(x_k))_i|$ is maximal. Here e_i with $1 \leq i \leq n$ denotes the i-th standard basis vector in $\mathbb{R}^n$.

a) Show that the direction p_k is a descent direction whenever x_k is no stationary point of f.

b) Assume that f is twice continuously differentiable and coercive and that the step lengths α_k satisfy the Wolfe conditions (see Nocedal & Wright, equation (3.6)). Show that $\lim_{k\to\infty} \|\nabla f(x_k)\| = 0$.