Exercise 1

- Let us consider inactive constraints first

$\forall i \in \mathcal{I} \setminus \mathcal{A}(x)$: $\quad a_i^T x - b_i =: \delta_i > 0$

$\forall p \in \mathbb{R}^n$ and $\forall \delta \in [0, \frac{\delta_i}{\|a_i\| \|p\|})$ positive number (possibly infinity)

$$a_i^T(x+\delta p) = a_i^T x + \delta a_i^T p \geq a_i^T x - \delta \|a_i\| \|p\|$$

$$\geq a_i^T x - \delta_i = b_i$$

Thus any direction is feasible with respect to inactive constaints.

- Let us consider equality constraints now.

For any $\delta > 0$: $\quad a_i^T(x+\delta p) = a_i^T x + \delta a_i^T p$

$$= b_i + \delta a_i^T p = b_i$$

$i \in \mathcal{E}$

<u>if and only if</u> $\quad a_i^T p = 0 \quad , i \in \mathcal{E}$

- Similarly, for active inequality constraints:

For any $\delta > 0$: $\quad a_i^T (x + \delta p) = a_i^T x + \delta a_i^T p$

$$= b_i + \delta a_i^T p \geq b_i, \; i \in \mathcal{I} \cap \mathcal{A}(x)$$

<u>if and only if</u> $\quad a_i^T p \geq 0, \; i \in \mathcal{I} \cap \mathcal{A}(x)$

Therefore, the direction $p \in \mathbb{R}^n$ is feasible for linear constraints

if and only if

$$a_i^T p = 0, \quad i \in \mathcal{E}$$

$$a_i^T p \geq 0, \quad i \in \mathcal{I} \cap \mathcal{A}(x)$$

Exercise 2

Put $\Omega = \{ p \in \mathbb{R}^n \mid \|p\| \leq \Delta_k \}$

Clearly $\Pi_\Omega(x) = \begin{cases} x, & \|x\| \leq \Delta_k \\ \Delta_k \dfrac{x}{\|x\|}, & \|x\| > \Delta_k \end{cases}$

Suppose that p is a locally optimal solution to the trust-region problem.

Then $\Pi_\Omega[p - \alpha \nabla m_k(p)] = p, \quad \forall \alpha > 0$

by (5).

If $\|p - \alpha \nabla m_k(p)\| \leq \Delta_k$ for some $\alpha > 0$

$\Rightarrow \Pi_\Omega[p - \alpha \nabla m_k(p)] = p - \alpha \nabla m_k(p) = p$

for this $\alpha > 0 \Rightarrow \nabla m_k(p) = 0.$

Therefore, we can take $\lambda = 0$ and get

$$\nabla m_k(p) = B_k p + g_k = 0$$

$$B_k p = -g_k$$

$$\begin{cases} (B_k + \lambda I) p = -g_k \\ \lambda(\|p\| - \Delta_k) = 0 \end{cases} \quad (\text{Because } \lambda \geq 0)$$

If $\|p - \alpha \nabla m_k(p)\| > \Delta_k$, $\forall \alpha > 0$

$$\Rightarrow \quad \Delta_k \frac{p - \alpha \nabla m_k(p)}{\|p - \alpha \nabla m_k(p)\|} = p \quad , \forall \alpha > 0$$

$$\nabla m_k(p) = -\alpha^{-1}\left(\frac{\|p - \alpha \nabla m_k(p)\|}{\Delta_k} - 1 \right) p$$

$$= -\lambda_\alpha p$$

Observe that $\lambda_\alpha > 0$ (we assumed that $\|p - \alpha \nabla m_k(p)\| > \Delta_k$). In fact, since $\nabla m_k(p)$ and p do not depend on α

$$\Rightarrow \quad \nabla m_k(p) = -\lambda p \quad , \text{for some } \lambda > 0.$$

Therefore

$$B_k p + g_k = -\lambda p$$

$$\begin{cases} (B_k + \lambda I)p = -g_k \quad , \lambda > 0 \\ \lambda(\|p\| - \Delta_k) = \\ \lambda\left(\Delta_k \dfrac{\|p - \alpha \nabla m_k(p)\|}{\|p - \alpha \nabla m_k(p)\|} - \Delta_k\right) = 0 \end{cases}$$

Exersise 3

Ω - closed, convex, non-empty

$z \in \mathbb{R}^n \setminus \Omega$

Consider the optimality conditions (6)

for the projection problem:

$x^* \in \Omega$:

$$(x^* - z)^T (x - x^*) \geq 0 \qquad \forall x \in \Omega$$

Put $a = x^* - z \neq 0$ ($x^* \in \Omega$, $z \notin \Omega$, Ω is closed)

$\forall x \in \Omega$:

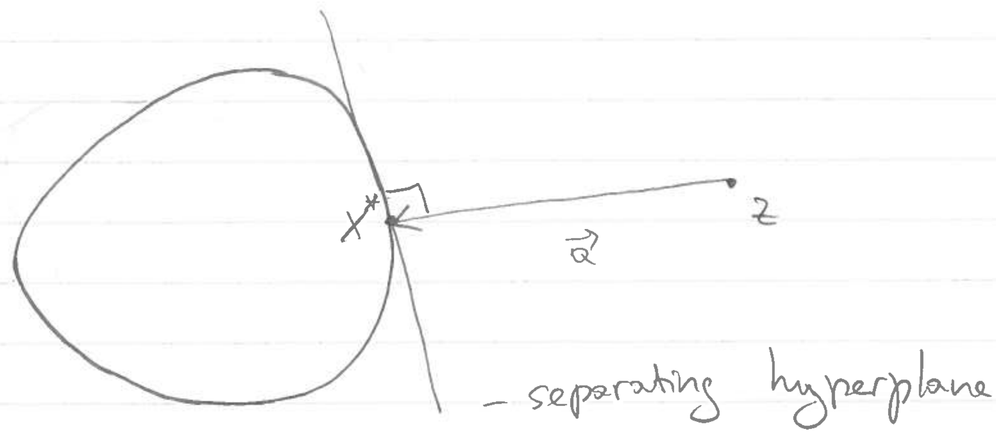
$$a^T x \geq a^T x^*$$

Also: $\|x^* - z\|_2^2 = (x^* - z)^T (x^* - z) > 0$

$$\Rightarrow \quad (x^* - z)^T x^* > (x^* - z)^T z$$

$$a^T x^* > a^T z \quad \Rightarrow$$

Thus we can take $b := a^T x^*$ ☒



Exercise 4

Ω - convex , $\bar{x} \in \Omega$

- $\vec{0}^T \cdot (x - \bar{x}) = 0 \leq 0 \quad \forall x \in \Omega$

  $\Rightarrow \vec{0} \in N_\Omega[\bar{x}] \Rightarrow N_\Omega[\bar{x}]$ is non-empty

- $q_1, q_2 \in N_\Omega[\bar{x}]$

$0 \leq \lambda \leq 1$

$q = \lambda q_1 + (1-\lambda) q_2$

$$q^T(x-\bar{x}) = \underbrace{\lambda}_{\geq 0} \underbrace{q_1^T(x-\bar{x})}_{\leq 0} + \underbrace{(1-\lambda)}_{\geq 0} \underbrace{q_2^T(x-\bar{x})}_{\leq 0} \leq 0$$

$\Rightarrow q \in N_\Omega(\bar{x}) \Rightarrow N_\Omega(\bar{x})$ is convex

- $q \in N_\Omega(\bar{x})$, $\alpha > 0$:

$$\underbrace{\alpha}_{>0} \underbrace{q^T(x-\bar{x})}_{\leq 0} \leq 0 \Rightarrow \alpha q \in N_\Omega(\bar{x})$$

$\Rightarrow N_\Omega(\bar{x})$ is a cone.

- Suppose that $\{q_i\} \to \bar{q}$, $q_i \in N_\Omega(\bar{x})$

Then $\bar{q}^T(x-\bar{x}) = \lim_{i\to\infty} \underbrace{q_i^T(x-\bar{x})}_{\leq 0} \leq 0$

$\Rightarrow \bar{q} \in N_\Omega(\bar{x}) \Rightarrow N_\Omega(\bar{x})$ is closed $\blacksquare$