TMA4180
Solutions to recommended exercises in Chapter 13 of N&W

Exercise 13.1

We consider the problem

$$\max_{x,y} c^T x + d^T y, \quad \text{subject to } A_1 x = b_1, A_2 x + B_2 y \leq b_2, l \leq y \leq u, \tag{1}$$

and want to rewrite it in standard form

$$\min_{z} e^T z, \quad \text{subject to } Az = b, z \geq 0. \tag{2}$$

First, we turn (1) into a minimization problem,

$$\max_{x,y} c^T x + d^T y = \min_{x,y} -c^T x + -d^T y.$$

The first constraint is already an equality constraint, so we keep it for now. For the other constraints, we define slack variables $r, s, t \geq 0$, so that the constraints can be written as

$$\begin{aligned} A_1 x &= b_1, \\ A_2 x + B_2 y + r &= b_2, \\ y - s &= l, \\ y + t &= u. \end{aligned}$$

The next trick is to split x and y into nonnegative and nonpositive parts,

$$\begin{aligned} x &= x^+ - x^-, \quad \text{where } x^+ = \max(x, 0) \geq 0, x^- = \max(-x, 0) \geq 0, \\ y &= y^+ - y^-, \quad \text{where } y^+ = \max(y, 0) \geq 0, y^- = \max(-y, 0) \geq 0. \end{aligned}$$

We can now write our original system (1) in standard form (2) with

$$z = \begin{bmatrix} x^+ \\ x^- \\ y^+ \\ y^- \\ r \\ s \\ t \end{bmatrix}, \quad e = \begin{bmatrix} -c \\ c \\ -d \\ d \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & -A_1 & 0 & 0 & 0 & 0 & 0 \\ A_2 & -A_2 & B_2 & -B_2 & I & 0 & 0 \\ 0 & 0 & I & -I & 0 & -I & 0 \\ 0 & 0 & I & -I & 0 & 0 & I \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ l \\ u \end{bmatrix}.$$

Exercise 13.5

We are given the problem

$$\min c^T x, \quad \text{subject to } Ax \geq b, x \geq 0, \tag{3}$$

and want to show that its dual is

$$\max b^T \lambda, \quad \text{subject to } A^T \lambda \geq c, \lambda \geq 0. \tag{4}$$

It suffices to show that the KKT systems are identical. We first write out the KKT conditions for the primal problem (3),

$$c - A^T \lambda - s = 0 \tag{5}$$
$$Ax \geq b \tag{6}$$
$$x \geq 0 \tag{7}$$
$$\lambda \geq 0 \tag{8}$$
$$s \geq 0 \tag{9}$$
$$\lambda^T (Ax - b) = 0 \tag{10}$$
$$s^T x = 0. \tag{11}$$

where the Lagrangian multiplier vector for the first and second constraint is λ and s respectively. Then, we reformulate the dual to a minimization problem,

$$\min -b^T \lambda \quad \text{subject to } c - A^T \lambda \geq 0, \lambda \geq 0,$$

Now, we use x and z as the Lagrangian multiplier vectors for the first and second constraints respectively. The KKT conditions are then:

$$-b + Ax - z = 0 \tag{12}$$
$$c - A^T \lambda \geq 0 \tag{13}$$
$$\lambda \geq 0 \tag{14}$$
$$x \geq 0 \tag{15}$$
$$z \geq 0 \tag{16}$$
$$x^T (c - A^T \lambda) = 0 \tag{17}$$
$$z^T \lambda = 0. \tag{18}$$

We observe that $s = c - A^T \lambda$ is a slack variable ($s \geq 0$) for the first dual constraint. From this, (5) follows immediately. Further, (6) follows from (12) and (16), while (7), (8) and (9) follows trivially from (15), (14) and (13) respectively. Combining (12) and (18) gives (10), while (17) gives (11).

Hence, we have shown that the KKT system for the primal follows from the KKT system for the dual. To go the other way, observe that $z = Ax - b$ is a slack variable for the primal problem. A similar procedure as above will give you the KKT conditions for the dual from the KKT conditions for the primal. You should verify this!

Representation theorem

We are considering the problem

$$\min_{x\in\Omega} c^T x, \quad \text{with } \Omega = \{-x_1 \le 0, -x_2 \le 0, -x_1 - x_2 \le -1\}.$$

Following to the note, we formulate Ω in the following way:

$$\Omega = \left\{x \in \mathbb{R}^2 | \tilde{A}x \le \tilde{b}\right\},$$

where

$$\tilde{A} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -1 & -1 \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

We observe that the only points $v_i \in \Omega$ s.t. rank $\bar{\bar{A}}_{v_i} = 2$, are $v_1 = (1,0)$ and $v_2 = (0,1)$, since these are the only points in Ω where we have two active constraints. The sets P and C are thus given as

$$\begin{aligned} P &= \{(\lambda_1, \lambda_2) | \lambda_1, \lambda_2 \ge 0, \lambda_1 + \lambda_2 = 1\}, \\ C &= \left\{d \in \mathbb{R}^2 | d_1, d_2 \ge 0\right\}. \end{aligned}$$

P is simply the straight line segment between v_1 and v_2, while C is the first quadrant. See Figure 1 for a drawing of Ω, P and C. The Representation theorem states that $\Omega = P + C$. From the drawing this seems intuitively correct. We will start with the point $x = (2,2)$ and show by the algorithm in the proof of Proposition 1, that it can be written as a sum $x = p + c$ where $p \in P$ and $c \in C$.

Since $\bar{\bar{A}}_x = \mathbf{0}$, $z_1 = (1,0) \in \text{Null } \bar{\bar{A}}_x$. It is easy to see that $x + \lambda z_1 \in \Omega$ for small λ. Also z_1 is in the first quadrant, so $z_1 \in C$. Next, we calculate λ^+ and λ^-,

$$\begin{aligned} \lambda^+ &= \sup\{\lambda \in \mathbb{R}^2 : x + \lambda z_1 \in \Omega\} = +\infty, \\ \lambda^- &= \sup\{\lambda \in \mathbb{R}^2 : x - \lambda z_1 \in \Omega\} = 2. \end{aligned}$$

Thus, we are in case 3, and take $x^- = x - \lambda^- z_1 = (0,2) \in \Omega$. Observe that $\bar{\bar{A}}_{x^-} = [-1, 0]$, and thus rank $\bar{\bar{A}}_{x^-} = 1$.

In the next iteration, we set $x = x^- = (0,2)$. Thus $\bar{\bar{A}}_x = [-1, 0]$, and $z_2 = (0,1) \in \text{Null } \bar{\bar{A}}_x$. Similarly as before, $x + \lambda z_2 \in \Omega$ for small λ, and $z_2 \in C$. Further,

$$\begin{aligned} \lambda^+ &= \sup\{\lambda \in \mathbb{R}^2 : x + \lambda z_2 \in \Omega\} = +\infty, \\ \lambda^- &= \sup\{\lambda \in \mathbb{R}^2 : x - \lambda z_2 \in \Omega\} = 1. \end{aligned}$$

We are again in case 3, and take $x^- = x - \lambda^- z_2 = (0,1) \in \Omega$. Observe that $\bar{\bar{A}}_{x^-} = \left[\begin{smallmatrix} -1 & 0 \\ -1 & -1 \end{smallmatrix}\right]$, so rank $\bar{\bar{A}}_{x^-} = 2$. Thus, $x^- = v_2 \in P$.

To sum up, we have

$$x = \underbrace{2z_1 + z_2}_{\in C} + \underbrace{v_2}_{\in P} .$$

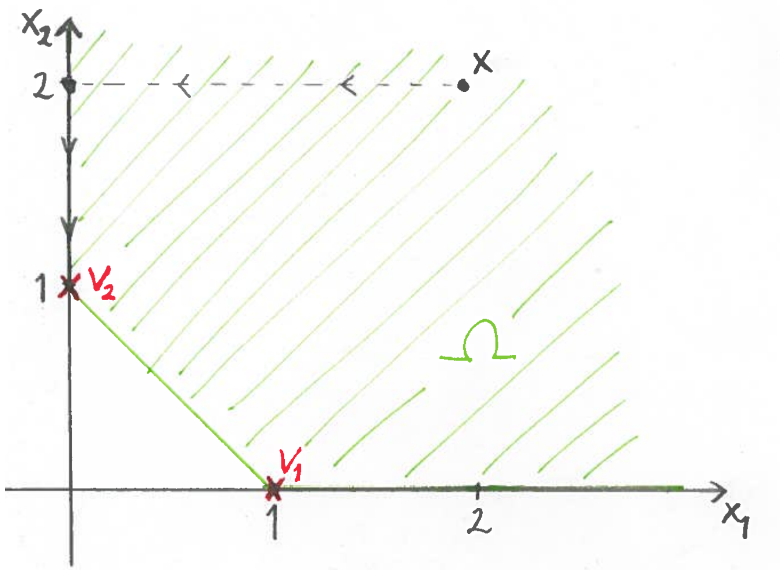

Figure 1: Schematic drawing of Ω. The vectors v_1 and v_2 are also shown, together with the point $x = (2, 2)$ and the path followed by the algorithm. The set C is simply the whole first quadrant.