TMA4180
Solutions to recommended exercises in Chapter 12 of N&W

Exercise 12.4

We consider the minimization problem

$$\min_{x\in\Omega} f(x), \tag{1}$$

where f and Ω are convex. We first show that local solutions are also global solutions. Let x_0 be any local solution. Then there exists a neighborhood $N(x_0)$ such that

$$f(x_0) \leq f(x^*), \qquad x \in N(x_0) \cap \Omega.$$

This proof is based on contradiction. Suppose x_0 is not a global solution. Then for the global solution, x^*, $f(x^*) < f(x_0)$. Since Ω is convex, there exists an $\alpha \in [0.1]$ such that

$$\alpha x_0 + (1-\alpha)x^* \in N(x_0) \cap \Omega.$$

Further, by convexity of f,

$$\begin{aligned} f(\alpha x_0 + (1-\alpha)x^*) &\leq \alpha f(x_0) + (1-\alpha) f(x^*) \\ &< f(x_0) + (1-\alpha) f(x_0) = f(x_0). \end{aligned}$$

This contradicts that x_0 is a minimum in $N(x_0) \cap \Omega$, and hence x_0 must be a global minimum.

Next, we prove that the set of all global solutions, S, is convex. Consider any two distinct points[1] $x_1, x_2 \in S$ and let $x = \alpha x_1 + (1-\alpha)x_2$ for some $\alpha \in (0,1)$. We need to prove that $x \in S$. By convexity of Ω, $x \in \Omega$, and by convexity of f,

$$\begin{aligned} f(\alpha x_1 + (1-\alpha)x_2) &\leq \alpha f(x_1) + (1-\alpha) f(x_2) \\ &= \alpha f(x_1) + (1-\alpha) f(x_1) = f(x_1), \end{aligned}$$

since $f(x_1) = f(x_2)$ (global solutions). The above inequality must be an equality, or else x_1 is not a global minimum. Thus, $x \in S$, and S is convex.

Exercise 12.15

At $x^* = (0,1)^T$, c_1 and c_2 are the active constraints, and hence $\lambda_3 = \lambda_4 = 0$. The KKT conditions at x^* are now

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= 2\left(x_1^* - \frac{3}{2}\right) + \lambda_1 + \lambda_2 = 0, \\ \frac{\partial \mathcal{L}}{\partial x_2} &= 4(x_2^* - t)^3 + \lambda_1 - \lambda_2 = 0, \\ &\lambda_1, \lambda_2 \geq 0. \end{aligned}$$

[1] Otherwise, if S consists of only one point, then S is trivially convex.

Solving for λ_1 and λ_2 in the two first equations give

$$\lambda_1 = 1 - \lambda_2,$$
$$\lambda_2 = \frac{1 - 4t^3}{2}.$$

Restricting the Lagrange multiplicators to be non-negative gives that $|t| \leq 4^{-1/3}$.

Now, set $t = 1$. The global minimum of f is $\left(\frac{3}{2}, 1\right)$. By strictly convexity of f and Ω, it follows that only the first constraint is active (convince yourself by an illustration). Hence, $\lambda_2 = \lambda_3 = \lambda_4 = 0$, and the KKT conditions reduces to

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} = 2\left(x_1 - \frac{3}{2}\right) + \lambda_1 = 0, \\
\frac{\partial \mathcal{L}}{\partial x_2} = 4(x_2 - 1)^3 + \lambda_1 = 0, \\
1 - x_1 - x_2 = 0, \\
\lambda_1 \geq 0,
\end{aligned}$$

whose solution is $(x_1, x_2) = (0.728, 0.272)$. By strictly convexity of f and Ω, this is a global solution.

Exercise 12.17

The KKT condition (12.34a) in N&W can be written as

$$\nabla f(x^*) - \sum_{i \in \mathcal{A}(x^*)} \lambda_i^* \nabla c_i(x^*) = 0$$
$$\sum_{i \in \mathcal{A}(x^*)} \lambda_i^* \nabla c_i(x^*) = \nabla f(x^*). \tag{2}$$

By LICQ, $\{\nabla c_i(x^*), i \in \mathcal{A}(x^*)\}$ is linearly independent, and hence (2) has a unique solution for $\lambda_i^*, i \in \mathcal{A}(x^*)$. Further, by (12.34e) in N&W, $\lambda_i = 0$, for $i \in \mathcal{I} \setminus \mathcal{A}(x^*)$.

Exercise 12.18

The KKT conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} = 2(x^* - 1) - 2(x^* - 1)\lambda^* = 0, \\
\frac{\partial \mathcal{L}}{\partial y} = 2(y^* - 2) + 5\lambda^* = 0, \\
(x^* - 1)^2 - 5y^* = 0.
\end{aligned}$$

The only solution to this system is $x^* = 1$, $y^* = 0$ and $\lambda^* = \frac{4}{5}$. Since $\nabla c_1(x) = (2(x-1), -5)^T \neq 0$, the LICQ is satisfied.

To show that this is a solution, we use the Second-Order Sufficent Conditions (Theorem 12.6 in N&W). From Eq. (12.55) in N&W,

$$w \in \mathcal{C}(x^*, y^*, \lambda^*) \Rightarrow w^T \nabla c_1(x^*, y^*) = 0.$$

Thus, $w_2 = 0$, and for all $w = (w_1, 0)$ with $w_1 \neq 0$,

$$w^T \nabla^2 \mathcal{L}(x^*, y^*, \lambda^*) w = \frac{2}{5} w_1^2 > 0.$$

Hence, $(1, 0)$ is a solution.

Now, if we substitute the constraint into the objective function, we get the unconstrained one dimensional minimization problem

$$\min g(y) = 5y + (y-2)^2 = \left(y + \frac{1}{2}\right)^2 + \frac{15}{4}.$$

This problem has the solution $\tilde{y} = -\frac{1}{2}$, but

$$g\left(-\frac{1}{2}\right) = \frac{15}{4} < 4 = f(x^*, y^*),$$

so the solution to this problem can not yield solutions for the original problem. In fact, $\tilde{y}$ can't be on on the parabola $5y = (x-1)^2$.

Why doesn't the two minimization problems yield the same solutions? By the constraint $5y = (x-1)^2$, it is implicitly given that $y \geq 0$, but this is lost when we consider the unconstrained problem $\min g(y)$. If we restrict $y \geq 0$, then the minimum of $g(y) = \left(y + \frac{1}{2}\right)^2 + \frac{15}{4}$ is indeed $y = 0$, because $g'(y) > 0$ for $y > 0$.

Exercise 12.19

For this exercise, we only give the correct answers, so you should do the computations yourself and verify that these are correct.

(a) Yes.

(b) Yes (with $\lambda_1^* = 2/3$ and $\lambda_2^* = 5/3$).

(c) $\mathcal{F}(x^*) = \{d \in \mathbb{R}^2 | -3d_1 - d_2 \geq 0, d_2 \geq 0\}$, $\mathcal{C}(x^*, \lambda^*) = \{0\}$.

(d) Yes (trivially) and yes (nothing to check).

Exercise 12.20

A contour plot of $f(x) = x_1 x_2$ is plotted in Figure 1 together with the unit circle. Obviously, both $(-2^{-\frac{1}{2}}, 2^{-\frac{1}{2}})^T$ and $(-2^{-\frac{1}{2}}, 2^{-\frac{1}{2}})^T$ are solutions of the minimization problem. It can be shown that all points $(\pm 2^{-\frac{1}{2}}, \pm 2^{-\frac{1}{2}})$ satisfy the KKT conditions, but that only $(-2^{-\frac{1}{2}}, 2^{-\frac{1}{2}})^T$ and $(-2^{-\frac{1}{2}}, 2^{-\frac{1}{2}})^T$ satisfy the second order conditions (the two other points are maximums).

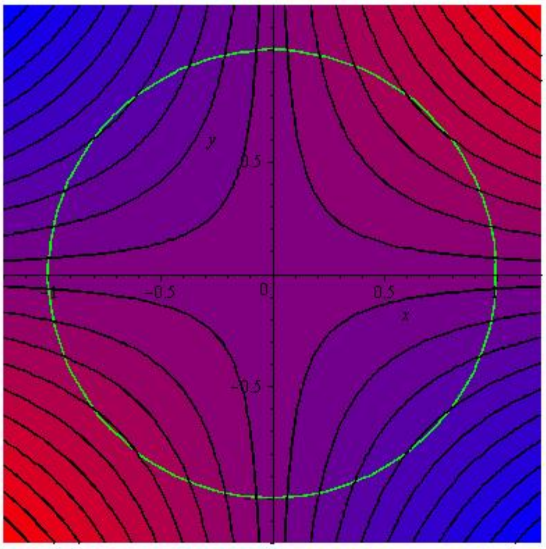

Figure 1: Contour plot of f.