Problem set 8, TMA4175

Karl Yngve Lervåg

March, 2008

Problem 1. *(Ex 10.16 from Nevanlinna)*
Let the analytic function $w(z)$ map a domain G_z one-to-one conformally onto a domain G_w; both domains have a well-defined area. Show that the area A of G_w is given by the double integral

$$A = \iint_{G_z} |w'(z)|^2 \, dxdy.$$

We have that

$$dA_w = dudv = \left| \frac{\partial(u,v)}{\partial(x,y)} \right| dxdy,$$

and since the mapping is conformal, $|\partial(u,v)/\partial(x,y)| = |w'(z)|^2$, hence

$$A_w = \iint_{G_w} dudv = \iint_{G_z} |w'(z)|^2 dxdy.$$

Problem 2. *(Ex 10.17 from Nevanlinna)*
The Bieberbach area theorem. *If*

$$w(z) = \sum_{n=0}^{\infty} c_n z^n$$

is analytic and schlicht (conform and bijectiv) in the disk $|z| \leq r$, then the area of the image domain is given by

$$A = \pi \sum_{n=1}^{\infty} n|c_n|^2 r^{2n}.$$

The derivative of w is

$$w'(z) = \sum_{n=1}^{\infty} nc_n z^{n-1},$$

thus

$$|w'(z)|^2 = w'(z)\overline{w'(z)} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} nmc_n\bar{c}_m z^{n-1}\bar{z}^{\,m-1}.$$

We can use this with the result from the previous exercise to obtain the wanted formula:

$$\begin{aligned}
A &= \iint_{|z|<r} |w'(z)|^2 dxdy \\
&= \sum_{n=1}\sum_{m=1} nmc_n\bar{c}_m \iint_{|z|<r} z^{n-1}\bar{z}^{\,m-1} dxdy \\
&= \sum_{n=1}\sum_{m=1} nmc_n\bar{c}_m \iint_{\substack{\rho<r \\ 0\leq\theta<2\pi}} \rho^{n+m-1}\ e^{i(n-m)\theta} d\rho d\theta \\
&= \sum_{n=1}\sum_{m=1} nmc_n\bar{c}_m \int_0^{2\pi} \Big(\cos([n-m]\theta) + i\sin([n-m]\theta)\Big) d\theta \int_0^r \rho^{n+m-1} d\rho \\
&= \sum_{n=1}\sum_{m=1} nmc_n\bar{c}_m\ 2\pi\delta_{nm}\frac{r^{n+m}}{n+m} \\
&= \sum_{n=1} n^2|c_n|^2\frac{2\pi}{2n}r^{2n} \\
&= \pi\sum_{n=1} n|c_n|^2\, r^{2n}.
\end{aligned}$$

Problem 3. *(Ex 10.18 from Nevanlinna)*
In addition to the assumptions made in the preceding exercise we assume that $w'(0) = 1$. Show that the area of the image domain is then greater than or equal to the area of the disk $|z| \leq r$. Under what condition does equality hold?

Assuming that $w'(0) = 1$ is equivalent to letting $c_1 = 1$, and thus

$$A = \pi r^2 + \pi \sum_{n=2}^{\infty} n|c_n|^2 r^{2n} > \pi r^2,$$

which proves what is asked. Equality holds when $c_k = 0$ for all $k > 1$, that is for

$$w(z) = C + z,$$

where C is a constant.

Problem 4. *(Ex 10.19 from Nevanlinna)*
Suppose that $w(z)$ is analytic in a domain G, that γ_z is a piecewise regular curve in G, and that $w'(z) \neq 0$ for $z \in \gamma_z$. Show that the length of the image curve γ_w of γ_z is

$$L = \int_{\gamma_z} |w'(z)|\,|dz|.$$

We have that

$$dw = w'(z)dz,$$

hence the length of the curve γ_w is

$$L = \int_{\gamma_w} |dw| = \int_{\gamma_z} |w'(z)||dz|.$$

Problem 5. *(Ex 11.3 from Nevanlinna)*
Find an analytic function $w(z)$ in the disk $|z - 1| < 1$ whose real part is $\log \sqrt{x^2 + y^2}$.

Let

$$w(z) = \overline{\log} z = \log \sqrt{x^2 + y^2} + i\theta,$$

where $-\pi/2 < \theta < \pi/2$.

Problem 6. *(Ex 11.4 from Nevanlinna)*
Prove that the angle subtended at a point z of the half-plane $Im\ z > 0$ by a given segment of the real axis is a harmonic function of z.

Let z be a point in the upper half-plane $Im\ z > 0$ and let a and b be two points on the real axis. Let α and β be the angles between the real axis and the lines from a and b to z, respectively, i.e.

$$\alpha = \arg(z - a),$$
$$\beta = \arg(z - b).$$

Let ω denote the angle subtended at the point z by the segment ab on the real axis, then

$$\omega(z) = \beta - \alpha = \arg\left(\frac{z-b}{z-a}\right).$$

Let $f(z) = (z - b)/(z - a)$, and

$$\log f(z) = \log|f(z)| + i \arg f(z).$$

Since $f(z)$ and $\log f(z)$ are analytic in the upper half-plane, $\omega(z) = \arg f(z)$ is harmonic.