Problem set 4, TMA4175

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Problem 1. *(Ex 7.4 from Nevanlinna)*
Determine the domain of convergence and the sum of the following geometric series:

(a) $-\sum_{k=1}^{\infty}\left(\frac{-1}{z+1}\right)^k$

(b) $\sum_{k=1}^{\infty}\left(\frac{z}{z-1}\right)^k$

(c) $\sum_{k=1}^{\infty}(z^2-1)^k$

(a) The series converges when

$$\left|\frac{1}{z+1}\right| < 1 \Leftrightarrow |z+1| > 1,$$

and the sum is

$$\begin{aligned}
-\sum_{k=1}^{\infty}\left(\frac{-1}{z+1}\right)^k &= 1 - \sum_{k=0}^{\infty}\left(\frac{-1}{z+1}\right)^k \\
&= 1 - \frac{1}{1-\left(\frac{-1}{z+1}\right)} \\
&= 1 - \frac{z+1}{z+2} \\
&= \frac{1}{z+2}.
\end{aligned}$$

(b) The series converges when

$$\left|\frac{z}{z-1}\right| < 1 \Leftrightarrow |z| < |z-1|.$$

This is the same as $Re(z) < 1/2$. The sum is

$$\begin{aligned}\sum_{k=1}^{\infty}\left(\frac{z}{z-1}\right)^k &= \sum_{k=0}^{\infty}\left(\frac{z}{z-1}\right)^k - 1 \\ &= \frac{1}{1-\left(\frac{z}{z-1}\right)} - 1 \\ &= -z.\end{aligned}$$

(c) The series converges when

$$|z^2-1| < 1 \Leftrightarrow |z-1||z+1| < 1.$$

In polar coordinates this expression simplifies to

$$|r^2e^{2i\theta}-1| < 1 \Leftrightarrow r^2 < 4\cos(2\theta),$$

which is a Lemniscate. The sum is

$$\begin{aligned}\sum_{k=1}^{\infty}(z^2-1)^k &= \sum_{k=0}^{\infty}(z^2-1)^k - 1 \\ &= \frac{1}{1-(z^2-1)} - 1 \\ &= \frac{1}{2-z^2} - 1 \\ &= \frac{z^2-1}{2-z^2}.\end{aligned}$$

Problem 2. *(Ex 7.8 from Nevanlinna)*
Set $z = r(\cos\phi + i\sin\phi)$ *in the series*

$$\sum_{n=0}^{\infty} z^n,$$

and split each term into its real and imaginary parts. Form the series of real parts and the series of imaginary parts, and find the sum of each when it converges.

Every term z^n can be split to $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$, which we use to find that

$$\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} r^n \cos(n\theta) + i\sum_{n=0}^{\infty} r^n \sin(n\theta).$$

The sum of a geometric series is known. It can be split into its real and imaginary parts,

$$\begin{aligned}
\sum_{k=0}^{\infty} z^n &= \frac{1}{1-z} \\
&= \frac{1-\overline{z}}{(1-z)(1-\overline{z})} \\
&= \frac{1-\overline{z}}{1+|z|^2-(z+\overline{z})} \\
&= \frac{1-r\cos\theta+ir\sin\theta}{1+r^2-2r\cos\theta} \\
&= \frac{1-r\cos\theta}{1+r^2-2r\cos\theta} + i\left(\frac{r\sin\theta}{1+r^2-2r\cos\theta}\right),
\end{aligned}$$

and hence

$$\begin{aligned}
\sum_{n=0}^{\infty} r^n \cos(n\theta) &= \frac{1-r\cos\theta}{1+r^2-2r\cos\theta} \\
\sum_{n=0}^{\infty} r^n \sin(n\theta) &= \left(\frac{r\sin\theta}{1+r^2-2r\cos\theta}\right).
\end{aligned}$$

Problem 3. *(Ex 7.9 from Nevanlinna)*
What can be said about the domain of convergence of the following power series?

$$\sum_{n=0}^{\infty} \frac{a_n}{(z-z_0)^n}$$

Let $f(\xi)$ be a general power series,

$$f(\xi) = \sum_{n=0}^{\infty} a_n (\xi - \xi_0)^n,$$

with convergence radius R, i.e.

$$|\xi - \xi_0| < R.$$

Now let $z = z_0 + \frac{1}{\xi - \xi_0}$ and let

$$g(z) = f\left(\xi_0 + \frac{1}{z-z_0}\right) = \sum_{n=0}^{\infty} \frac{a_n}{z-z_0}.$$

The domain of convergence of the last series becomes

$$|z - z_0| = \frac{1}{|\xi - \xi_0|} > \frac{1}{R}.$$

Problem 4. *(Ex 7.11 from Nevanlinna)*
Determine the radius of convergence of the following power series:

(a) $\sum n^p z^n$

(b) $\sum \frac{z^n}{n^n}$

(c) $\sum q^{n^2} z^n \quad (|q| < 1)$

(d) $\sum \left(\frac{z}{2+(-1)^n}\right)^n$

(a)

$$R = \underline{\lim}_{n\to\infty} \frac{1}{\sqrt[n]{n^p}} = \lim_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{p}{n}} = 1^p = 1$$

(b)

$$R = \underline{\lim}_{n\to\infty} \frac{1}{\sqrt[n]{(1/n)^n}} = \lim_{n\to\infty} n = \infty$$

(c)

$$R = \underline{\lim}_{n\to\infty} \frac{1}{\sqrt[n]{q^{n^2}}} = \lim_{n\to\infty} \frac{1}{q^n} = \infty$$

(d)

$$R = \underline{\lim}_{n\to\infty} \frac{1}{\sqrt[n]{\frac{1}{2+(-1)^n}}} = \lim_{n\to\infty} \sqrt[n]{1} = 1$$

Problem 5. *(Ex 7.12 from Nevanlinna)*
Determine the region of convergence of the following series:

(a) $\sum \frac{(-1)^n}{z+n}$

(b) $\sum \frac{z^n}{1-z^n}$

(c) $\sum \frac{z^n}{z^{2n}+1}$

(a) We start by considering the series

$$S = \sum_{n=0}^{\infty} \left(\frac{1}{z+2n} - \frac{1}{z+2n+1} \right) = \sum_{n=0}^{\infty} \frac{1}{(z+2n)(z+2n+1)}.$$

This series converges absolutely for all $z \neq -2n, -(2n+1)$, because

$$\left| \frac{1}{z+2n} \cdot \frac{1}{z+2n+1} \right| \leq C(z) \frac{1}{4n^2} \sim \frac{1}{n^2}.$$

Now

$$\sum_{n=0}^{2n} \frac{(-1)^n}{z+n} \to S,$$
$$\sum_{n=0}^{2n+1} \frac{(-1)^n}{z+n} \to S,$$

because

$$\frac{(-1)^{2n+1}}{z+2n+1} \to 0.$$

Since both the even and the odd sums are convergent and have the same sum, the power series is convergent and has the same sum.

(b) First, we see that $z^n \neq 1$. Consider first $|z| < 1$. Then

$$\left| \frac{z^n}{1-z^n} \right| \leq \frac{|z|^n}{1-|z|^n} \leq \frac{|z|^n}{1-|z|},$$

and the series is dominated by a convergent power series and is hence absolutely convergent. Now consider $|z| > 1$, then

$$\left| \frac{z^n}{1-z^n} \right| \geq \frac{|z|^n}{1+|z|^n} \geq \frac{|z|^n}{2|z|^n} = \frac{1}{2},$$

and the series is clearly divergent, since the nth term does not approach zero.

(c) We have that $z^{2n} \neq -1$. When $|z| = 1$ we see that

$$\left|\frac{z^n}{1+z^{2n}}\right| \geq \frac{|z|^n}{1+|z|^{2n}} = \frac{1}{2},$$

and the series is divergent. When $|z| < 1$ we get

$$\left|\frac{z^n}{1+z^{2n}}\right| \leq \frac{|z|^n}{1-|z|^{2n}} \leq \frac{|z|^n}{1-|z|^2},$$

which shows that the series converges absolutely, since the geometric series does. At last, when $|z| > 1$ and n is large enough,

$$\left|\frac{z^n}{1+z^{2n}}\right| \leq \frac{|z|^n}{|z|^{2n}-1} \leq \frac{|z|^n}{\frac{1}{2}|z|^{2n}} = \frac{2}{|z|^n}.$$

The dominating geometric series is absolutely convergent, thus so is the given series.

Problem 6. *(Ex 7.15 from Nevanlinna)*
Prove that if $|a_n/a_{n+1}|$ tends to a finite limit ρ as n tends to infinity, then ρ is the radius of convergence of the series $\sum a_n z^n$.

Assume $|a_n/a_{n+1}| \to \rho$, and consider

$$\left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right| \to \frac{|z|}{\rho} = L.$$

According to the ratio test for series with positive terms, the series converges absolutely when $L < 1$, i.e.

$$|z| < \rho,$$

and diverges when $L > 1$. Thus ρ is the radius of convergence.

Problem 7. *(Ex 8.6 from Nevanlinna)*
Prove the following theorem (from section 8.7 in Nevanlinna):

If $w_1(z), w_2(z), \ldots$ are continuous functions defined on a rectifiable curve l, and if the series

$$w(z) = \sum_{\nu=1}^{\infty} w_\nu(z)$$

converges uniformely on this curve, then the series can be integrated term by term:

$$\int_l w(z)dz = \sum_{\nu=1}^{\infty} \int_l w_\nu(z)dz.$$

Problem 8. *(Ex 8.7 from Nevanlinna)*
Evaluate

$$I = \int_{-i}^{i} |z| dz$$

for the following paths of integration:

(*a*) *A straigth-line segment*

(*b*) *The arc* $|z| = 1, Re \geq 0$.

(*c*) *The arc* $|z| = 1, Re \leq 0$.

(a) Letting $z = x + iy$ and $x = 0$ gives $z = iy$ and $dz = idy$, hence

$$I = \int_{-1}^{1} i|y|dy = 2i \int_{0}^{1} ydy = i.$$

(b) A simple parametrization of the arc $|z| = 1$ is to use polar coordinates. We get $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta$ and $-\pi/2 < \theta < \pi/2$, resulting in

$$I = \int_{-\pi/2}^{\pi/2} ie^{i\theta} d\theta = 2i.$$

(c) We use the same approach as before, this time with $3\pi/2 > \theta > \pi/2$, hence

$$I = \int_{3\pi/2}^{\pi/2} ie^{i\theta} d\theta = -\int_{\pi/2}^{3\pi/2} ie^{i\theta} d\theta = 2i.$$