Problem set 1, TMA4175

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Problem 1. *Let*

$$w = \frac{a-b}{1-\overline{a}b}.$$

(*a*) *Prove that* $|w| = 1$ *if* a *and* b *are complex numbers with* $|a| = 1$ *or* $|b| = 1$.

(*b*) *Prove that* $|w| < 1$ *if both* $|a|$ *and* $|b|$ *are less than 1.*

(*c*) *Prove that*

$$\left| \frac{az+b}{\overline{b}z+\overline{a}} \right| = 1,$$

for $|z| = 1$.

(a) First, we have that

$$|b-a|^2 = |b|^2 + |a|^2 - (b\overline{a} + \overline{b}a),$$
$$|1-\overline{a}b|^2 = 1 + |a|^2|b|^2 - (b\overline{a} + \overline{b}a).$$

Now we see that

$$|w| = 1$$
$$\Updownarrow$$
$$|b-a|^2 = |1-\overline{a}b|^2$$
$$\Updownarrow$$
$$|b|^2 + |a|^2 = 1 + |a|^2|b|^2,$$

and the equality holds when either $|a|$ or $|b|$ is 1.

(b) Using the same approach we get that

$$\begin{gathered}
|w| < 1 \\
\Updownarrow \\
|b|^2 + |a|^2 = 1 + |a|^2|b|^2 \\
\Updownarrow \\
|b|^2(1 - |a|^2) < 1 - |a|^2, \\
\Updownarrow \\
|b|^2 c < c,
\end{gathered}$$

where c is positive when $|a| < 1$. Thus one can divide both sides with c, and we end up with $|b| < 1$, which proves that the claim holds.

(c) Again we can use the same approach. Write

$$\begin{aligned}
|az + b|^2 &= |a|^2|z|^2 + |b|^2 + (a\overline{b}z + \overline{a}bz), \\
|\overline{b}z + \overline{a}|^2 &= |b|^2|z|^2 + |a|^2 + (a\overline{b}z + \overline{a}bz),
\end{aligned}$$

and as before,

$$\begin{gathered}
\left|\frac{az + b}{\overline{b}z + \overline{a}}\right| = 1 \\
\Updownarrow \\
|az + b|^2 = |\overline{b}z + \overline{a}|^2 \\
\Updownarrow \\
|a|^2|z|^2 + |b|^2 = |b|^2|z|^2 + |a|^2,
\end{gathered}$$

and the last equality holds trivially when $|z| = 1$.

Problem 2. *Separate the following expressions into real and imaginary parts (z is complex):*

(a) $f(z) = iz^3$

(b) $f(z) = \frac{1}{z-i}$

(c) $f(z) = \frac{z-i}{z+i}$

(d) $f(z) = \frac{1}{z^2}$

This is equivalent to finding $f(z) = f(x, y) = u(x, y) + iv(x, y)$.

(a) By using that $z = x + iy$ we find

$$\begin{aligned} iz^3 &= u(x,y) + iv(x,y) \\ &= y^3 - 3x^2y \quad + \quad i(x^3 - 3xy^2). \end{aligned}$$

(b)

$$\begin{aligned} \frac{1}{z-i} &= u(x,y) + iv(x,y) \\ &= \frac{x}{x^2 + (1-y)^2} \quad + \quad i\left(\frac{1-y}{x^2 + (1-y)^2}\right). \end{aligned}$$

(c)

$$\begin{aligned} \frac{z-i}{z+i} &= u(x,y) + iv(x,y) \\ &= \frac{x^2 + y^2 - 1}{x^2 + (y+1)^2} \quad + \quad i\left(\frac{-2}{x^2 + (1-y)^2}\right). \end{aligned}$$

(d)

$$\begin{aligned} \frac{1}{z^2} &= u(x,y) + iv(x,y) \\ &= \frac{x^2 - y^2}{(x^2 + y^2)^2} \quad + \quad i\left(\frac{-2xy}{(x^2 + y^2)^2}\right). \end{aligned}$$

Problem 3. *Prove that the real and imaginary parts of an analytic function $f(z) = u(x,y) + iv(x,y)$ satisfy Laplace's equation,*

$$\Delta U = U_{xx} + U_{yy} = 0,$$

provided u and v possess continuous partial derivatives of second order.

We have that

$$\begin{aligned}\Delta u &= u_{xx} + u_{yy} \\ &= v_{yx} - v_{xy} \\ &= 0\end{aligned}$$

by using the Cauchy-Riemann equations, $u_x = v_y$ and $u_y = -v_x$. The last equality follows from the fact that the second order partial derivatives are continuous. Δv can be checked in the same manner.

Problem 4. *What is the most general polynomial, with real coefficients, of the form*

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3$$

which is the real part of an analytic function? Construct this function.

We have that $u(x,y) = ax^3 + 3bx^2y + 3cxy^2 + dy^3$, and that

$$\begin{aligned}\Delta u &= 0 \\ &\Downarrow \\ a &= -c, \\ b &= -d,\end{aligned}$$

giving $u(x,y) = ax^3 + 3bx^2y - 3axy^2 - by^3$. By using the Cauchy-Riemann equations we find that $v(x,y) = 3ax^2y + 3bxy^2 - ay^3 - bx^3 + C$, where C is a constant. To find the function $f(z)$ we can use that $x = (z+\overline{z})/2$ and $y = -i(z-\overline{z})/2$. An easier way is to write

$$\begin{aligned}f(z) &= a(x^3 - 3xy^2 + i(3x^2y - y^3)) - ib(x^3 - 3xy^2 + i(y^3 - 3x^2y)) + iC \\ &= az^3 - ibz^3 \\ &= (a - ib)z^3,\end{aligned}$$

where the second equality can be found by using the result in problem 2 (a).

Problem 5. *Suppose that the function $w(z)$ is analytic in a domain G which is symmetric with respect to the real axis. Show that $f(z) = \overline{w(\overline{z})}$ is then an analytic funtion of z in G.*

This problem can be solved by using the definition of the derivative. We have that

$$\begin{aligned}
\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} &= \lim_{\Delta z \to 0} \frac{\overline{w(\overline{z + \Delta z})} - \overline{w(\overline{z})}}{\Delta z} \\
&= \lim_{\overline{\Delta z} \to 0} \overline{\left[\frac{w(\overline{z} + \overline{\Delta z}) - w(\overline{z})}{\overline{\Delta z}} \right]} \\
&= \overline{w'(z)},
\end{aligned}$$

since w is analytic at $\overline{z}$ and $\Delta z \to 0 \Rightarrow \overline{\Delta z} \to 0$. Another approach is to use the Cauchy-Riemann equations and the fact that when $w(z) = u(x, y) + iv(x, y)$, then $w(\overline{z}) = u(x, -y) + if(x, -y)$. The proof is omitted.