# Complex analysis with potential theory

Xu Wang

ABSTRACT. These are the course notes for the course TMA4175 [2024] "Complex Analysis" at NTNU. It is mainly based on the complex analysis book [A0] of Ahlfors and will be updated after each lecture.

CONTENTS



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## 1. Basic complex analysis

### 1.1. Holomorphic functions.

**Definition 1.** *A set $U \subset \mathbb{C}$ is said to be* ***open*** *if for every $z \in U$,*

$$\mathbb{D}_z(r) := \{w \in \mathbb{C} : |w - z| < r\} \subset U$$

*for some $r > 0$.*

*Example*: The unit disk (centered at the origin)

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$$

is open. The *annulus*

$$\mathbb{D}_{r,1} := \{z \in \mathbb{C} : r < |z| < 1\}, \ \ 0 < r < 1,$$

is also open.

**Definition 2.** *A smooth function $f$ on an open set $U \subset \mathbb{C}$ is said to be* ***holomorphic*** *("analytic" in the Ahlfors book) if it satisfies the* ***Cauchy–Riemann equation***

$$\frac{\partial f}{\partial \bar{z}} = 0, \ \ \text{on } U.$$

*(Our definition is different from page 69, Definition 10 in the Ahlfors book, but in any case, they are equivalent, see page 122 for smoothness of the Ahlfors analytic functions.)*

**Remark**: *Write $z = x + iy \in \mathbb{C}$, then*

$$x = \frac{z + \bar{z}}{2}, \ \ y = \frac{z - \bar{z}}{2i}, \ \ \frac{\partial x}{\partial \bar{z}} = \frac{1}{2}, \ \ \frac{\partial y}{\partial \bar{z}} = \frac{-1}{2i} = \frac{i}{2}, \tag{1.1}$$

*thus*

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{i}{2}\frac{\partial f}{\partial y}. \tag{1.2}$$

*Exercise 1*: For smooth $f(x+iy) = u(x+iy) + iv(x+iy)$ show that the followings are equivalent:

(1) $f$ is holomorphic;

(2) $u_x = v_y$ and $v_x = -u_y$.

**Solution**: A direct computation gives

$$f_{\bar{z}} = \frac{1}{2}(u_x - v_y) + \frac{i}{2}(v_x + u_y),$$

hence (1) and (2) are equivalent.

*Exercise 2*: Let $f = u + iv$ be holomorphic. Show that $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$.

**Solution**: By Exercise 1, we have $u_{xx} = v_{xy}$ and $v_{xx} = -u_{xy}$, thus $u_{xx} + u_{yy} = 0$. Similar proof for $v_{xx} + v_{yy} = 0$.

*Exercise 3*: For smooth $f(x+iy) = u(x+iy) + iv(x+iy)$, we define the complex Jacobian and real Jacobian as

$$\mathrm{Jac}^{\mathbb{C}}(f) := \det \begin{pmatrix} f_z & f_{\bar{z}} \\ (\bar{f})_z & (\bar{f})_{\bar{z}} \end{pmatrix}, \quad \mathrm{Jac}(f) := \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

Show that $\mathrm{Jac}^{\mathbb{C}}(f) = \mathrm{Jac}(f)$.

**Solution**: Note that

$$\mathrm{Jac}^{\mathbb{C}}(f) = |f_z|^2 - |f_{\bar{z}}|^2.$$

Input

$$f_{\bar{z}} = \frac{1}{2}(u_x - v_y) + \frac{i}{2}(v_x + u_y), \; f_z = \frac{1}{2}(u_x + v_y) + \frac{i}{2}(v_x - u_y),$$

we obtain

$$\mathrm{Jac}^{\mathbb{C}}(f) = u_x v_y - u_y v_x = \mathrm{Jac}(f).$$

**Proposition 1** (See Ahlfors page 22-27)**.** *Let $f$ be an holomorphic function on an open set $U \subset \mathbb{C}$. Then the* ***complex derivative***

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \tag{1.3}$$

*exists and satisfies*

$$f' = \frac{\partial f}{\partial z} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{i}{2}\frac{\partial f}{\partial y} \tag{1.4}$$

*on $U$, moreover, we have*

$$\frac{df(g(s))}{ds} = f'(g(s))\frac{dg(s)}{ds} \tag{1.5}$$

*for every smooth function $g : (a,b) \to U$, where $a < b$ are real numbers.*

*Proof.* From the taylor expansion of $f$ with respect to $z, \bar z$

$$f(z+h) = f(z) + h\frac{\partial f}{\partial z} + \bar h \frac{\partial f}{\partial \bar z} + O(|h|^2)$$

and $\frac{\partial f}{\partial \bar z} = 0$, we get

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h\to 0} \frac{h\frac{\partial f}{\partial z} + O(|h|^2)}{h} = \frac{\partial f}{\partial z},$$

thus the complex derivative $f'(z)$ exists and equals $\frac{\partial f}{\partial z}$. Similar to the proof of (1.2), we have

$$\frac{\partial f}{\partial z} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{i}{2}\frac{\partial f}{\partial y}$$

thus (1.4) follows. (1.5) follows from the chain rule

$$\frac{df(g(s))}{ds} = \frac{\partial f}{\partial z}|_{z=g(s)} \frac{dg(s)}{ds} + \frac{\partial f}{\partial \bar z}|_{z=g(s)} \frac{d\overline{g(s)}}{ds} = \frac{\partial f}{\partial z}|_{z=g(s)} \frac{dg(s)}{ds}$$

and (1.4). □

1.2. **Cauchy integral theorem.** Let us think of a *piecewise smooth closed curve* in $\mathbb{C}$ as a *piecewise smooth $2\pi$-periodic function* $\gamma : \mathbb{R} \to \mathbb{C}$ (i.e. $\gamma$ is piecewise smooth, $\gamma(t+2\pi) = \gamma(t)$ for every $t \in \mathbb{R}$ and the associated closed curve, as a set, is given by the image of $\gamma$). More precisely, the *curve associated to* $\gamma$ is given by the following set

$$\{\gamma(t) : t \in [0, 2\pi)\}$$

with *orientation* from $0$ to $2\pi$. The fundamental theorem in complex analysis is the following:

**Theorem 1** (Cauchy integral theorem). *Let $f$ be a holomorphic function on an open set, say $U$, in $\mathbb{C}$. Assume that inside $U$ we can deform a piecewise smooth closed curve $\gamma_0$ piecewise smoothly to another piecewise smooth closed curve $\gamma_1$. Then*

$$\int_{\gamma_0} f(z)\, dz = \int_{\gamma_1} f(z)\, dz.$$

*Proof.* We shall only prove the smooth case and leave the piecewise smooth case to the readers. Let $\{\gamma_s\}_{0\le s\le 1}$ be a smooth family in $U$ joining $\gamma_0$ and $\gamma_1$. Put

$$F(s) := \int_{\gamma_s} f(z)\, dz = \int_0^{2\pi} f(\gamma_s(t))\, d(\gamma_s(t)) = \int_0^{2\pi} f(\gamma_s)\gamma_s'\, dt,$$

where $\gamma_s'$ denotes the $t$-derivative of $\gamma_s$. It suffices to show that $F$ does not depend on $s$, or equivalently $dF(s)/ds = 0$ for all $s \in [0,1]$. Compute (8th Jan)

$$\frac{dF(s)}{ds} = \int_0^{2\pi} \frac{d(f(\gamma_s)\gamma_s')}{ds}\, dt = \int_0^{2\pi} \frac{d(f(\gamma_s))}{ds}\gamma_s' + f(\gamma_s)\frac{d(\gamma_s')}{ds}\, dt.$$

By (1.5), we have

$$\frac{d(f(\gamma_s))}{ds} = f'(\gamma_s)\frac{d\gamma_s}{ds},$$

thus

$$\frac{dF(s)}{ds} = \int_0^{2\pi} f'(\gamma_s)\gamma_s' \frac{d\gamma_s}{ds} + f(\gamma_s)\frac{d(\gamma_s')}{ds}\, dt.$$

Notice that

$$f'(\gamma_s)\gamma_s' \frac{d\gamma_s}{ds} + f(\gamma_s)\frac{d(\gamma_s')}{ds} = \frac{d}{dt}\left(f(\gamma_s)\frac{d\gamma_s}{ds}\right),$$

hence

$$\frac{dF(s)}{ds} = \int_0^{2\pi} \frac{d}{dt}\left(f(\gamma_s)\frac{d\gamma_s}{ds}\right) dt = \left(f(\gamma_s)\frac{d\gamma_s}{ds}\right)|_{t=2\pi} - \left(f(\gamma_s)\frac{d\gamma_s}{ds}\right)|_{t=0}.$$

Now, since each $\gamma_s$ is $2\pi$-periodic in $t$, we know the right hand side of the above equality vanishes. The proof is complete. □

**Remark**: *The precise meaning for, $\{\gamma_s\}_{0\leq s\leq 1}$ is a smooth family in $U$ joining $\gamma_0$ and $\gamma_1$, is the following: there exists a smooth function*

$$\gamma : (s,t) \mapsto \gamma(s,t) \in U,$$

*on a neighborhood of $(s,t) \in [0,1] \times \mathbb{R}$ such that each*

$$\gamma_s : t \mapsto \gamma(s,t)$$

*is $2\pi$-periodic and*

$$\gamma(0,t) = \gamma_0(t), \ \ \gamma(1,t) = \gamma_1(t).$$

*One may similarly define the piecewise smooth case.*

**Definition 3.** *Let $U$ be an open set in $\mathbb{C}$. Two piecewise smooth closed curves in $U$ are said to be* ***$U$-homotopic*** *to each other if there exists a piecewise smooth family in $U$ joining them. We write*

$$\gamma_0 \sim_U \gamma_1$$

*if $\gamma_0$ is $U$-homologous to $\gamma_1$. In case $\gamma_1$ is a single point, we shall write $\gamma_0 \sim_U 0$ and say that $\gamma_0$ is* ***$U$-homotopic to zero (or $U$-contractible)****.*

Now one may rephrase the Cauchy integral theorem as follows.

**Theorem 2** (Cauchy integral theorem-homotopic version)**.** *Let $f$ be a holomorphic function on an open set, say $U$, in $\mathbb{C}$. If $\gamma_0 \sim_U \gamma_1$ then*

$$\int_{\gamma_0} f(z)\, dz = \int_{\gamma_1} f(z)\, dz,$$

*in particular,*

$$\int_\gamma f(z)\, dz = 0$$

*if $\gamma \sim_U 0$.*

### 1.3. Simply connected domains.

**Definition 4.** *An open set $\Omega \subset \mathbb{C}$ is called a* ***domain (region)*** *if any two points in $\Omega$ can be connected by a piecewise smooth curve in $\Omega$.*

**Remark**: *In case $\Omega$ is a domain, one may check that $\sim_\Omega$ in Definition 3 is an equivalence relation. Our definition of a domain is different from the Ahlfors definition in page 57, Definition 4. But in any case, they are equivalent by the following theorem.*

**Theorem 3** (Page 56, Theorem 3)**.** *Let $\Omega$ be a nonempty open set in $\mathbb{C}$. Then the followings are equivalent:*

(1) *$\Omega$ is a domain;*
(2) *If $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1$, $\Omega_2$ open and $\Omega_1 \cap \Omega_2 = \emptyset$ then either $\Omega_1$ or $\Omega_2$ is empty;*
(3) *Any two points in $\Omega$ can be connected by a piecewise horizontal/vertical curve in $\Omega$.*

*Proof. (1) implies (2)*: Otherwise, both are nonempty, so we can choose $p \in \Omega_1$, $q \in \Omega_2$ and connect $p, q$ with a piecewise smooth curve $\gamma(t)$ in $\Omega$ with $\gamma(0) = p$, $\gamma(1) = q$. Consider

$$T := \sup\{0 \leq t \leq 1 : \gamma(t) \in \Omega_1\},$$

Since $\gamma(1) \notin \Omega_1$, we know that $0 < T < 1$. If $\gamma(T) \in \Omega_1$, then the continuity of $\gamma$ implies that $\gamma(T + \varepsilon) \in \Omega_1$ for some $\varepsilon > 0$, this of course can not happen by the maximum property of $T$. So we must have $\gamma(T) \in \Omega_2$, then continuity of $\gamma$ implies that $\gamma(T - \varepsilon, T + \varepsilon) \subset \Omega_2$, this can not happen either. So we know that one of $\Omega_1$, $\Omega_2$ must be empty.

*(2) implies (3)*: If $p, q$ in $\Omega$ can not be connected by a piecewise horizontal/vertical curve in $\Omega$, then one may define $\Omega_1$ to be the collection of points in $\Omega$ that connects to $p$ by a piecewise horizontal/vertical curve in $\Omega$. Then the complement, say $\Omega_2$, of $\Omega_1$ is precisely the collection of those points that can not be connected to $p$ by a piecewise horizontal/vertical curve in $\Omega$. The basic observation is that $\Omega_1$ and $\Omega_2$ are disjoint nonempty open, which contradicts (2).

*(3) implies (1)*: directly from the definition. □

**Definition 5.** *A domain $\Omega \subset \mathbb{C}$ is said to be* ***simply connected*** *if $\gamma \sim_\Omega 0$ for every piecewise smooth closed curve $\gamma$ in $\Omega$. (9th Jan)*

**Remark**: *Our definition is different from the one given by Ahlfors in page 139, where $\Omega$ is said to be simply connected if its complement with respect to the extended plane $\mathbb{C} \cup \{\infty\}$ is connected.*

**Theorem 4** (See page 141, Corollary 1)**.** *If $f$ is holomorphic in a simply connected domain $\Omega \subset \mathbb{C}$ then*

$$\int_\gamma f(z)\, dz = 0$$

*for every piecewise smooth closed curve $\gamma$ in $\Omega$.*

*Example*: Convex open sets are simply connected. The *annulus*

$$\mathbb{D}_{r,1} := \{z \in \mathbb{C} : r < |z| < 1\}, \ \ 0 < r < 1,$$

are not simply connected, since

$$\int_{|z|=\frac{r+1}{2}} \frac{dz}{z} = 2\pi i \neq 0.$$

Theorem 4 also implies the following result.

**Corollary 1** (Page 142, Corollary 2 in the Ahlfors book)**.** *If $f(z)$ is holomorphic and $\neq 0$ in a simply connected domain $\Omega \subset \mathbb{C}$ then it is possible to define single-valued analytic branches of $\log f(z)$ and $(f(z))^{1/n}$ in $\Omega$.*

*Proof.* Fix $z_0 \in \Omega$, by Theorem 4, we know that

$$F(z) := \int_{z_0}^{z} \frac{f'(w)}{f(w)}\, dw$$

does not depend on the choice of piecewise smooth curves connecting $z_0$ and $z$ and we have

$$F'(z) = \frac{f'(z)}{f(z)}.$$

Thus $f(z)e^{-F(z)}$ has the derivative zero and is therefore a constant. Choosing one of the infinitely many values $\log f(z_0)$, we find that

$$e^{F(z)-F(z_0)+\log f(z_0)} = f(z),$$

and consequently we can set

$$\log f(z) = F(z) - F(z_0) + \log f(z_0).$$

To define $(f(z))^{1/n}$, we merely write it in the form $e^{(1/n)\log f(z)}$. (15th Jan) ∎

1.4. **Cauchy's integral formula.** Let $f$ be a holomorphic function on an open set $U \in \mathbb{C}$. Fix $z \in U$. Assume that inside $U \setminus \{z\}$ we can deform a piecewise smooth closed curve $\gamma$ piecewise smoothly to $\gamma_\varepsilon : t \mapsto z + \varepsilon e^{it}$ for all sufficient small $\varepsilon > 0$. Then Theorem 1 implies that

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \lim_{\varepsilon \to 0+} \frac{1}{2\pi i} \int_{\gamma_\varepsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta = f(z). \tag{1.6}$$

In case $\gamma$ is given by $|\zeta - a| = r$ (always with the anti-clockwise orientation), then (1.6) gives:

**Theorem 5** (Cauchy's integral formula)**.** *Let $f$ be a holomorphic function on a neighborhood of the disk $|\zeta - a| \leq r$. Then*

$$\frac{1}{2\pi i} \int_{|\zeta-a|=r} \frac{f(\zeta)}{\zeta - z}\, d\zeta = f(z), \quad \forall\, z \text{ with } |z-a| < r. \tag{1.7}$$

**Corollary 2** (page 122, Ahlfors book)**.** *Bounded holomorphic functions on $\mathbb{C}$ are constants.*

*Proof.* Take the derivative of (1.7), we obtain

$$\frac{1}{2\pi i}\int_{|\zeta-a|=r}\frac{f(\zeta)}{(\zeta-z)^2}\,d\zeta=f'(z),\quad \forall\, z \text{ with } |z-a|<r.$$

Assume that $|f|\leq M$ on $\mathbb{C}$. Then the above formula gives

$$|f'(a)|=\Big|\frac{1}{2\pi i}\int_{|\zeta-a|=r}\frac{f(\zeta)}{(\zeta-a)^2}\,d\zeta\Big|\leq\frac{M}{r}.$$

Let $r$ tend to $\infty$, we get $f'(a)=0$ for every $a\in\mathbb{C}$. Thus $f$ is a constant. □

Liouville's theorem leads to a proof of the following fundamental theorem of algebra.

**Corollary 3** (page 122, Ahlfors book)**.** *Polynomials of positive degree in $\mathbb{C}[z]$ always have a zero point in $\mathbb{C}$.*

*Proof.* Let $P\in\mathbb{C}[z]$ be a polynomial of positive degree. If $P$ has no zero point then $P(z)^{-1}$ is holomorphic on $\mathbb{C}$ and tends to zero for $z\to\infty$. Thus $P(z)^{-1}$ is bounded and the Liouville theorem implies that it is a constant, thus $P$ is a constant, we get a contradiction. □

*Exercise 1*: Use (1.7) to prove the following results for holomorphic function $f$ defined on a neighborhood of the disk $|\zeta-a|\leq r$:

(1) For all positive integers $k$, we have

$$\frac{k!}{2\pi i}\int_{|\zeta-a|=r}\frac{f(\zeta)}{(\zeta-z)^{k+1}}\,d\zeta=f^{(k)}(z),\quad \forall\, z \text{ with } |z-a|<r. \tag{1.8}$$

*Solution*: Take $z$-derivatives of (1.7).

(2) For all $z$ with $|z-a|<r$, we can write

$$f(z)-f(a)=(z-a)f_1(z),$$

where $f_1$ is holomorphic on the whole disc $|z-a|<r$ and

$$f_1(z)=\frac{1}{2\pi i}\int_{|\zeta-a|=r}\frac{f(\zeta)}{(\zeta-z)(\zeta-a)}\,d\zeta$$

*Solution*: (1.7) also gives

$$f(z)-f(a)=\frac{1}{2\pi i}\int_{|\zeta-a|=r}\frac{f(\zeta)}{\zeta-z}-\frac{f(\zeta)}{\zeta-a}\,d\zeta=\frac{z-a}{2\pi i}\int_{|\zeta-a|=r}\frac{f(\zeta)}{(\zeta-z)(\zeta-a)}\,d\zeta,$$

which implies (2).

(3) Show that in case $f$ is a constant, (2) gives

$$\frac{1}{2\pi i}\int_{|\zeta-a|=r}\frac{1}{(\zeta-z)(\zeta-a)}\,d\zeta=0,\quad \forall\, a,z\in\mathbb{C} \text{ with } |z-a|<r. \tag{1.9}$$

*Solution*: Take $f=1$, then $1-1=(z-a)f_1(z)$ gives $f_1=0$.

(4) Show that (1.9) implies that for every positive integer $k$,

$$\frac{1}{2\pi i}\int_{|\zeta-a|=r}\frac{1}{(\zeta-z)(\zeta-a)^k}\,d\zeta=0,\quad \forall\, a,z\in\mathbb{C}\text{ with }|z-a|<r. \tag{1.10}$$

*Solution*: By the Cauchy integral theorem

$$\frac{1}{2\pi i}\int_{|\zeta-a|=r}\frac{1}{(\zeta-z)(\zeta-a)}\,d\zeta=\frac{1}{2\pi i}\int_{|\zeta|=R}\frac{1}{(\zeta-z)(\zeta-a)}\,d\zeta$$

for sufficiently big $R>0$. Hence (1.9) implies

$$0=\frac{1}{2\pi i}\int_{|\zeta|=R}\frac{1}{(\zeta-z)(\zeta-a)}\,d\zeta$$

for $R>0$ big enough. Apply the $a$-derivative of the above formula, we obtain

$$\frac{1}{2\pi i}\int_{|\zeta|=R}\frac{1}{(\zeta-z)(\zeta-a)^k}\,d\zeta$$

for every positive integer $k$. Hence the Cauchy integral theorem gives

$$\frac{1}{2\pi i}\int_{|\zeta-a|=r}\frac{1}{(\zeta-z)(\zeta-a)^k}\,d\zeta=\frac{1}{2\pi i}\int_{|\zeta|=R}\frac{1}{(\zeta-z)(\zeta-a)^k}\,d\zeta=0.$$

(5) Let us continue the process in (2) and write

$$f_1(z)-f_1(a)=(z-a)f_2(z),$$

use (1.10) to show that

$$f_2(z)=\frac{1}{2\pi i}\int_{|\zeta-a|=r}\frac{f_1(\zeta)}{(\zeta-z)(\zeta-a)}\,d\zeta=\frac{1}{2\pi i}\int_{|\zeta-a|=r}\frac{f(\zeta)}{(\zeta-z)(\zeta-a)^2}\,d\zeta. \tag{1.11}$$

*Solution*: Since $f_1(\zeta)=(f(\zeta)-f(a))/(\zeta-a)$, we have

$$\frac{1}{2\pi i}\int_{|\zeta-a|=r}\frac{f_1(\zeta)}{(\zeta-z)(\zeta-a)}\,d\zeta=\frac{1}{2\pi i}\int_{|\zeta-a|=r}\frac{f(\zeta)-f(a)}{(\zeta-z)(\zeta-a)^2}\,d\zeta,$$

which equals the right hand side of (1.11) by (1.10).

(6) Let us inductively define $f_{n+1}(z)$ such that $f_n(z)-f_n(a)=(z-a)f_{n+1}(z)$ show that (see page 125, Theorem 8 of the Ahlfors book)

$$f(z)=f(a)+f'(a)(z-a)+\cdots+\frac{f^{(n)}(a)}{n!}(z-a)^n+f_{n+1}(z)(z-a)^{n+1}, \tag{1.12}$$

with

$$f_{n+1}(z)=\frac{1}{2\pi i}\int_{|\zeta-a|=r}\frac{f(\zeta)}{(\zeta-a)^{n+1}(\zeta-z)}\,d\zeta \tag{1.13}$$

and

$$f^{(n)}(a)=\frac{n!}{2\pi i}\int_{|\zeta-a|=r}\frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta. \tag{1.14}$$

*Solution*: By induction on $n$, we have

$$f(z) = f(a) + f_1(a)(z-a) + \cdots + f_n(a)(z-a)^n + f_{n+1}(z)(z-a)^{n+1}. \tag{1.15}$$

Take $z = a$, we know that (1) implies (1.14), (1.13) implies $f_n(a) = \frac{f^{(n)}(a)}{n!}$. Hence it suffices to prove (1.13). Apply (1.7) to $f = f_{n+1}$, we obtain

$$f_{n+1}(z) = \frac{1}{2\pi i}\int_{|\zeta - a| = r} \frac{f_{n+1}(\zeta)}{(\zeta - z)}\, d\zeta. \tag{1.16}$$

By (1.15), we have

$$f_{n+1}(\zeta) = f(\zeta)(\zeta-a)^{-n-1} - f(a)(\zeta-a)^{-n-1} - f_1(a)(\zeta-a)^{-n} - \cdots - f_n(a)(\zeta-a)^{-1}.$$

Input it into (1.16), we know that (4) gives (1.13).

(7) Denote by $M$ the maximum of $|f|$ on the circle $|\zeta - a| = r$, use (1.13) to show that

$$|f_{n+1}(z)(z-a)^{n+1}| \leq \frac{M|z-a|^{n+1}}{r^n(r - |z-a|)}. \tag{1.17}$$

In particular, as $n \to \infty$, $f_{n+1}(z)(z-a)^{n+1}$ tends to zero uniformly in every smaller disk $|z - a| \leq \delta < r$ (this is the proof of Theorem 6 below in page 179 of the Ahlfors book). *Solution*: Follows from (write $\zeta - z = \zeta - a - (z-a)$)

$$|f_{n+1}(z)| \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{M}{r^{n+1}(r - |z-a|)} r\, d\theta = \frac{M}{r^n(r - |z-a|)}.$$

*Exercise 2*: Compute the following integrals:

(1) $\int_\gamma x\, dz$, where $\gamma$ is the directed line segment from $0$ to $1+i$; *Answer*: $(1+i)/2$.

(2) $\int_{|z|=r} x\, dz$; *Answer*: $i\pi r^2$.

(3) $\int_{|z|=2} \frac{dz}{z^2-1}$; *Answer*: 0.

(4) $\int_{|z|=1} |z-1| \cdot |dz|$. *Answer*: 8.

*Exercise 3*: Let $P(z)$ be a polynomial in $z$ and $C$ denote the circle $|z - a| = R$, show that

$$\int_C P(z)\, d\bar{z} = -2\pi i R^2 P'(a).$$

*Solution*: Note that on $C$ we have

$$\bar{z} - \bar{a} = \frac{R^2}{(z-a)},$$

which gives

$$d\bar{z} = -\frac{R^2}{(z-a)^2}\, dz$$

on $C$. Thus

$$\frac{1}{2\pi i}\int_C \frac{P(z)}{(z-a)^2}\, dz = P'(a)$$

gives our formula.

1.5. **Taylor and Laurent series.** The Cauchy integral formula implies that every holomorphic function can be locally written as a convergent power series (called Taylor series) of $z$, in particular, holomorphic functions are always real analytic. (16th Jan)

**Theorem 6.** *If $f$ is holomorphic on a domain $\Omega \subset \mathbb{C}$ then the representation*

$$f(z) = f(a) + f'(a)(z-a) + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \cdots \tag{1.18}$$

*is valid in the largest open disk of center $a$ contained in $\Omega$. Moreover, the series converges uniformly in every smaller disk.*

More generally, we have the Laurent series of holomorphic functions on an annulus.

**Theorem 7.** *If $f$ is holomorphic in an annulus $R_1 < |z-a| < R_2$ then $f$ can be developed in a Laurent series of the form*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n, \quad c_n = \frac{1}{2\pi i}\int_{|z-a|=r} \frac{f(z)\,dz}{(z-a)^{n+1}}, \quad \forall\, R_1 < r < R_2, \tag{1.19}$$

*which converges uniformly on every circle inside the annulus.*

*Proof.* See page 185 of the Ahlfors book for the proof. (22th Jan) □

1.6. **Zeros, residues, poles and local description of holomorphic mappings.**

1.6.1. *Zeros, residues and poles.* A direct consequence of the Taylor series expansion, Theorem 6, is the following

**Proposition 2** (Page 126, Ahlfors book)**.** *Let $f$ be a holomorphic function on a domain $\Omega \subset \mathbb{C}$. If for some $a \in \Omega$, $f(a) = 0$ and all derivatives $f^{(\nu)}(a)$ vanish, then $f$ is identically zero in $\Omega$.*

*Proof.* Let $E$ be the set on which $f$ and all derivatives vanish. We know that $\Omega \setminus E$ is open (try!). Theorem 6 implies that $E$ is also open. Since $\Omega$ is a domain, we must have $E = \Omega$. □

Assume that $f(z)$ is not identically zero. Then if $f(a) = 0$ then there exists a first derivative $f^{(h)}(a) \neq 0$, we say then that $a$ is an order $h$ zero of $f$. To summarize, we shall introduce the following definition.

**Definition 6.** *Let $f$ be a holomorphic function on a domain $\Omega \subset \mathbb{C}$. Assume that $f(a) = 0$ for some $a \in \Omega$. We say the $a$ is an **order** $h$ **zero** of $f$, $h \in \mathbb{Z}$, $h > 0$, if*

$$f^{(h)}(a) \neq 0 \text{ and } f^{(\nu)}(a) = 0, \quad \forall\, 0 \leq \nu < h.$$

*In this case we shall write $h = \mathrm{Ord}_a f$.*

**Remark**: *By* (1.12)*, we have*

$$f(z) = (z-a)^h f_h(z), \text{ with } h = \mathrm{Ord}_a f, \quad f_h(a) \neq 0. \tag{1.20}$$

*Since $f_h$ is continuous (in fact holomorphic), we know that $f_h \neq 0$ in a neighborhood of $a$. Thus $z = a$ is the only zero of $f$ in this neighborhood. We have proved the following result:*

**Proposition 3** (Page 127, Ahlfors book)**.** *The zeros of a holomorphic function which does not vanish identically are isolated. Moreover, let $f, g$ be holomorphic functions on a domain $\Omega \subset \mathbb{C}$, if $f = g$ on a set which has an accumulation point then $f$ is identically equal to $g$ in $\Omega$.*

We consider now a holomorphic function $f$ in a punctured disk

$$\mathbb{D}_{0,R} := \{z \in \mathbb{C} : 0 < |z - a| < R\}, \quad R > 0.$$

The point $a$ is called an isolated singularity of $f$. By Theorem 7 about Laurent series expansions $f$ can be written as

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

in $\mathbb{D}_{0,R}$. We shall follow Berndtsson's notes on residue calculus [B, Definition 1] and define:

**Definition 7.** *The coefficient $c_{-1}$ above is called the **residue** of $f$ at $a$. We write*

$$c_{-1} := \mathrm{Res}_a f.$$

*(One may compare with Definition 3, page 149 in the Ahlfors book).*

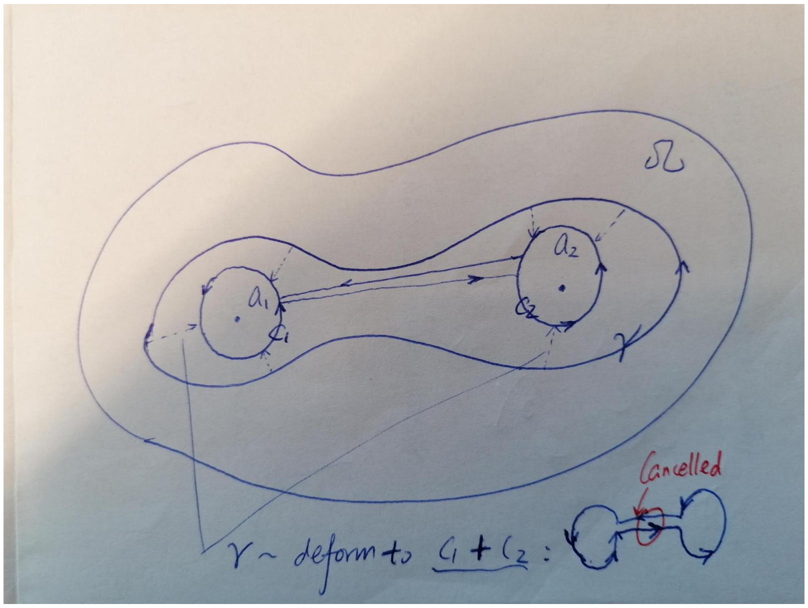
Theorem 7 (take $n = -1$ for the $c_n$ formula there) also gives the following theorem.

**Theorem 8** (Residue theorem, circle case)**.** *For $f$ holomorphic in $0 < |z - a| < R$ we have*

$$\frac{1}{2\pi i} \int_{|z-a|=r} f(z)\, dz = \mathrm{Res}_a f, \quad 0 < r < R.$$

To generalize the above theorem, it is convenient to introduce (23th Jan):

**Definition 8.** *Let $\gamma$ be a piecewise smooth closed curve in a domain $\Omega \subset \mathbb{C}$. Let $\{a_j\}_{1 \le j \le N}$ be $N$ points in $\Omega$, we say that $\gamma$ **encloses** $\{a_j\}_{1 \le j \le N}$ **in** $\Omega$ if $\gamma$ can be shrunk to $\{a_j\}_{1 \le j \le N}$ in $\Omega$, more precisely, it means that $\gamma$ is $\Omega \setminus \{a_j\}_{1 \le j \le N}$-homotopic to $\sum_{j=1}^{N} C_j$ for some small circle $C_j$ around $a_j$, see the picture below.*




Apply Theorem 1, Theorem 8 can be generalized to

**Theorem 9** (Residue theorem, homotopy version). *Let $\gamma$ be a piecewise smooth closed curve enclosing $\{a_j\}_{1\le j\le N}$ in a domain $\Omega\subset\mathbb{C}$. Then*

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz=\sum_{j=1}^N \mathrm{Res}_{a_j} f. \tag{1.21}$$

What we will usually use is the following special case of the above theorem.

**Theorem 10** (Residue theorem). *Let $f$ be a holomorphic function on $\Omega\setminus\{a_j\}_{1\le j\le N}$. Assume that there is a closed disc $D$ in $\Omega$ such that all $a_j$ lie in the interior of $D$. Then*

$$\frac{1}{2\pi i}\int_{\partial D} f(z)\,dz=\sum_{j=1}^N \mathrm{Res}_{a_j} f, \tag{1.22}$$

*for $f$ holomorphic in $\Omega\setminus\{a_j\}_{1\le j\le N}$.*

*Proof.* Use the above theorem and the fact that the curve $\partial D$ encloses $\{a_j\}_{1\le j\le N}$ in $\Omega$. □

**Remark**: The above result holds true for general $D$ with piecewise smooth boundary. One proof (optional in this course) is to use the Stokes theorem:

$$\frac{1}{2\pi i}\int_{\partial D} f\,dz=\frac{1}{2\pi i}\int_D d(f\,dz)=\frac{1}{2\pi i}\int_D \overline{\partial} f\wedge dz=\sum_{j=1}^N \mathrm{Res}_{a_j} f,$$

since

$$\overline{\partial}\left(\frac{1}{z-a_j}\right)\wedge dz=2\pi i\,\delta_{a_j},$$

where $\delta_{a_j}$ denotes the delta measure (function) at $a_j$.

*Exercise 1*: Compute $\int_{|z|=\rho} \frac{|dz|}{|z-a|^2}$ under the condition $|a| \neq \rho$. *Hint:* make use of the equations $z\bar{z} = \rho^2$ and $|dz| = -i\rho\frac{dz}{z}$.

*Solution*: By the hint, our integral becomes

$$\int_{|z|=\rho} \frac{-i\rho\frac{dz}{z}}{(z-a)(\rho^2/z-\bar{a})} = \int_{|z|=\rho} \frac{-i\rho dz}{(z-a)(\rho^2-\bar{a}z)}.$$

Let us apply the Cauchy integral theorem, if $|a| < \rho$, then we have

$$\int_{|z|=\rho} \frac{-i\rho dz}{(z-a)(\rho^2-\bar{a}z)} = \frac{2\pi\rho}{\rho^2-|a|^2};$$

if $|a| > \rho$, we have

$$\int_{|z|=\rho} \frac{-i\rho dz}{(z-a)(\rho^2-\bar{a}z)} = \frac{2\pi\rho}{|a|^2-\rho^2}.$$

Hence

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = \frac{2\pi\rho}{||a|^2-\rho^2|}.$$

*Exercise 2*: Compute ($n, m$ are integers)

$$\int_{|z|=1} e^z z^{-n}\, dz,\ \int_{|z|=2} z^n(1-z)^m\, dz.$$

*Solution*: By the residue theorem, we have

$$\int_{|z|=1} e^z z^{-n}\, dz = \begin{cases} 0 & n \leq 0 \\ 2\pi i & n = 1 \\ 2\pi i/(n-1)! & n \geq 2. \end{cases}$$

For the second integral, by the Cauchy integral theorem, we know that if $n, m \geq 0$, the integral is zero. If $n \geq 0$, $m = -1$ we have

$$\int_{|z|=2} z^n(z-a)^{-1}\, dz = 2\pi i a^n$$

for every $|a| < 2$, take the $a$-derivative, we get (for positive $k$)

$$\int_{|z|=2} z^n(z-a)^{-(k+1)}\, dz = 2\pi i \binom{n}{k} a^{n-k},$$

where we define $\binom{n}{k} = 0$ if $k > n$. Thus

$$\int_{|z|=2} z^n(1-z)^m\, dz = (-1)^m 2\pi i \binom{n}{-m-1}$$

for $n \geq 0$, $m < 0$. Similarly, we have

$$\int_{|z|=2} z^n(1-z)^m\,dz = (-1)^{n+1}2\pi i\binom{m}{-n-1}$$

for $m \geq 0$, $n < 0$ and

$$\int_{|z|=2} z^n(1-z)^m\,dz = 0$$

for $n, m < 0$.

*Exercise 3*: Find the poles and residues of the following functions:

(a) $\frac{1}{z^2+5z+6}$; *Solution*: $\mathrm{Res}_{-2} f = 1,\ \mathrm{Res}_{-3} f = -1.$

(b) $\frac{1}{(z^2-1)^2}$; *Solution*: $\mathrm{Res}_1 f = \frac{1}{4},\ \mathrm{Res}_{-1} f = \frac{1}{4}.$

(c) $\frac{1}{\sin z}$; *Solution*: $\mathrm{Res}_{2\pi\mathbb{Z}} f = 1,\ \mathrm{Res}_{\pi+2\pi\mathbb{Z}} f = -1.$

(d) $\cot z$; *Solution*: $\mathrm{Res}_{\pi\mathbb{Z}} f = 1.$

(e) $\frac{1}{\sin^2 z}$; *Solution*: $\mathrm{Res}_{\pi\mathbb{Z}} f = 0.$

Another notion of singularity is the pole order.

**Definition 9.** *Let $f$ be a holomorphic function on a punctured disk around $a$. We say the $a$ is an **order** $h$ **pole** of $f$, $h \in \mathbb{Z}$, $h > 0$, if its Laurent series reduces to*

$$f(z) = \sum_{n=-h}^{\infty} c_n(z-a)^n, \quad c_{-h} \neq 0.$$

*In this case we shall write $h = \mathrm{Ord}_a^P f$ ($P$ for pole). A function that is holomorphic except for poles is called a **meromorphic function**.*

**Remark**: *It is clear that $a$ is an order $h$ pole of $f$ if and only if $(z-a)^h f(z)$ extends to a holomorphic function, say $g$, in a neighborhood of $a$ with $g(a) \neq 0$. Hence we know that a function is meromorphic if and only if it is locally a quotient of two holomorphic functions.*

**Definition 10.** *Let $f$ be holomorphic on a punctured disk around $a$. We say the $a$ is a **removable singularity** of $f$ if its Laurent series has no negative terms ($c_n = 0$ for all $n < 0$). In case its Laurent series has infinitely many negative terms, we call $a$ an **essentially singularity** of $f$.*

The following theorem of Weierstrass gives a characterization of the behavior of a function around an essential singularity.

**Theorem 11** (Page 129, Theorem 9, Ahlfors)**.** *A holomorphic function comes arbitrarily close to any value in every neighborhood of an essential singularity.*

*Proof.* Otherwise, we can find a complex number $A$ and $\delta > 0$ such that $|f(z) - A| > \delta$ around $a$. Then $a$ must be a removable singularity of

$$g(z) := \frac{1}{f(z) - A},$$

(one may check that for $g$, all $c_n = 0$, $n < 0$), thus $g$ is holomorphic around $a$ and

$$f = \frac{1}{g} + A$$

is meromorphic near $a$. Hence $a$ can not be an essential singularity of $f$. (29th Jan) $\square$

*Exercise 1*: (a) Prove the following Green's formula for the unit square:

$$\int_{\partial R} p(x,y)\,dx + q(x,y)\,dy = \int_R (q_x - p_y)\,dxdy, \tag{1.23}$$

where $p, q$ are smooth function on a neighborhood of

$$R := \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1,\ 0 \le y \le 1\}.$$

*Solution*: Note that

$$\int_{\partial R} p\,dx = \int_0^1 p(x,0) - p(x,1)\,dx.$$

Hence the Newton-Lebniz formula

$$p(x,1) - p(x,0) = \int_0^1 p_y(x,y)\,dy$$

gives

$$\int_{\partial R} p\,dx = -\int_0^1 \left( \int_0^1 p_y(x,y)\,dy \right) dx = -\int_R p_y\,dxdy.$$

A similar argument gives $\int_{\partial R} q\,dy = \int_R q_x\,dxdy$. Hence (1.23) follows.

(b) Use (a) to prove that

$$\int_{\partial R} f\,dz = 2i \int_R f_{\bar{z}}\,dxdy, \tag{1.24}$$

where $f$ is smooth on a neighborhood of $R$.

*Solution*: Since $dz = dx + idy$, we have

$$f\,dz = f\,dx + if\,dy.$$

Apply (1.23) to $p = f,\ q = if$, we obtain

$$\int_{\partial R} f\,dz = \int_R (if)_x - f_y\,dxdy = 2i \int_R f_{\bar{z}}\,dxdy.$$

*Exercise 2*: Use the Residue theorem to verify the following integrals:

(a) $\int_0^\infty \frac{x^2\,dx}{x^4+5x^2+6} = \frac{\pi}{2}(\sqrt{3} - \sqrt{2})$;

*Solution*: Consider the square, say $D_R$, with vertices $(-R,0),\ (0,R),\ (R,iR),\ (-R,iR)$ and observe that

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{x^4+5x^2+6} = \lim_{R\to\infty} \int_{\partial D_R} \frac{z^2\,dz}{z^4+5z^2+6} = \lim_{R\to\infty} \int_{\partial D_R} \frac{z^2\,dz}{(z^2+2)(z^2+3)}.$$

Put

$$f(z)=\frac{z^2}{(z^2+2)(z^2+3)}.$$

We know that

$$\int_{\partial D_R} f(z)dz = 2\pi i\left(\mathrm{Res}_{\sqrt{2}i}f+\mathrm{Res}_{\sqrt{3}i}f\right)=(\sqrt{3}-\sqrt{2})\pi$$

for large $R$. Thus

$$\int_0^\infty \frac{x^2\,dx}{x^4+5x^2+6}=\frac{1}{2}\int_{-\infty}^\infty \frac{x^2\,dx}{x^4+5x^2+6}=\frac{\pi}{2}(\sqrt{3}-\sqrt{2}).$$

(b) $\int_0^\infty \frac{x\sin x\,dx}{x^2+1}=\frac{\pi}{2e}$;

*Solution*: Similarly, one may consider

$$f(z)=\frac{ze^{iz}}{z^2+1}$$

and observe that

$$\int_{-\infty}^\infty \frac{xe^{ix}\,dx}{x^2+1}=2\pi i\,\mathrm{Res}_i f=\frac{i\pi}{e}.$$

Compare the imaginary part, we obtain

$$\int_{-\infty}^\infty \frac{x\sin x\,dx}{x^2+1}=\frac{\pi}{e}.$$

Thus

$$\int_0^\infty \frac{x\sin x\,dx}{x^2+1}=\frac{1}{2}\int_{-\infty}^\infty \frac{x\sin x\,dx}{x^2+1}=\frac{\pi}{2e}.$$

(c) (hard) $\int_0^\infty \frac{x^{1/3}\,dx}{1+x^2}=\frac{\pi}{\sqrt{3}}$.

*Solution*: Can be found in this link (3) g.

*Exercise 3*: Let $f$ be a holomorphic function on a domain $\Omega\subset\mathbb{C}$. Assume that $|f(z)|=1$ for all $z\in\Omega$. Show that $f$ is a constant on $\Omega$.

*Solution*: If $f$ is not a constant, then the image of $f$ will be an open set by Corollary 4. But a non-empty open set can never be a subset of the unit circle.

1.6.2. *Argument principle.*

**Theorem 12** (Argument principle, circle version)**.** *Let $f$ be meromorphic on a neighborhood of the disk $|z-z_0|\le r$. Then $f$ has finite zeros, say $\{a_j\}_{1\le j\le N}$ and finite poles, say $\{b_k\}_{1\le k\le M}$ in that disk. Assume that all $a_j, b_k$ are away from the circle $|z-z_0|=r$, then*

$$\frac{1}{2\pi i}\int_{|z-z_0|=r}\frac{f'(z)}{f(z)}\,dz=\sum_{j=1}^N \mathrm{Ord}_{a_j}f-\sum_{k=1}^M \mathrm{Ord}^P_{b_k}f. \tag{1.25}$$

*Proof.* By Proposition 3, we know that $f$ has only finite zeros and poles in the disk. To prove (1.25), we shall write

$$f(z) = \frac{(z-a_1)^{\mathrm{Ord}_{a_1} f} \cdots (z-a_N)^{\mathrm{Ord}_{a_N} f}}{(z-b_1)^{\mathrm{Ord}^P_{a_1} f} \cdots (z-b_M)^{\mathrm{Ord}^P_{b_M} f}}\, g,$$

where $g$ is holomorphic and $\neq 0$ in $|z-z_0| \le r$. Forming the logarithmic derivative we obtain

$$\frac{f'(z)}{f(z)} = \sum_{j=1}^{N} \frac{\mathrm{Ord}_{a_j} f}{z-a_j} - \sum_{k=1}^{M} \frac{\mathrm{Ord}^P_{b_k} f}{z-b_k} + \frac{g'(z)}{g(z)}$$

for $z \neq a_j, b_k$, and particularly on the circle $|z-z_0| = r$, Since $g(z) \neq 0$ in the disk, we know that $g'/g$ is holomorphic around the disk, thus Theorem 2 yields

$$\int_{|z-z_0|=r} \frac{g'(z)}{g(z)}\, dz = 0,$$

together with

$$\int_{|z-z_0|=r} \frac{dz}{z-a_j} = \int_{|z-z_0|=r} \frac{dz}{z-b_k} = 2\pi i,$$

we obtain (1.25). □

Apply Theorem 1, Theorem 12 can be generalized to

**Theorem 13** (Argument principle, homotopy version)**.** *Let $f$ be meromorphic on a domain $\Omega \subset \mathbb{C}$. Let $\gamma$ be a piecewise smooth closed curve enclosing zeros $\{a_j\}_{1\le j\le N}$ and poles $\{b_k\}_{1\le k\le M}$ of $f$ in $\Omega$. Assume that $f$ has no zeros and poles on $\gamma$. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_{j=1}^{N} \mathrm{Ord}_{a_j} f - \sum_{k=1}^{M} \mathrm{Ord}^P_{b_k} f. \tag{1.26}$$

**Remark**: *The function $w = f(z)$ maps $\gamma$ onto a closed curve, say $\Gamma$, in the $w$-plane and we find*

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = \int_\gamma \frac{df}{f} = \int_\Gamma \frac{dw}{w}. \tag{1.27}$$

*If $\gamma$ is given by a $2\pi$ periodic function $\gamma(t)$, then $\Gamma$ is defined by $f(\gamma(t))$ and*

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = \int_\Gamma \frac{dw}{w} = \int_0^{2\pi} \frac{d\left(|f(\gamma(t)|e^{i\theta(f(\gamma(t))}\right)}{|f(\gamma(t)|e^{i\theta(f(\gamma(t))}} = \int_0^{2\pi} d\log|f(\gamma(t)| + i\int_0^{2\pi} d\theta(f(\gamma(t)).$$

*Note that $\int_0^{2\pi} d\log|f(\gamma(t)| = 0$, hence we have*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz = \frac{1}{2\pi}\int_0^{2\pi} d\theta(f(\gamma(t)).$$

*Thus the left hand side of (1.26) equals the average of change of argument of $f(z) \in \Gamma$ as $z$ traverses $\gamma$, this explains why the above theorem is referred to as the argument principle. The* *right hand side of* (1.26) *implies this average is always an integer (this integer is called the winding number of* $\gamma$ *with respect to the origin, see page 115 of the Ahlfors book).*

1.6.3. *Local description of holomorphic mappings.* In case $f$ is holomorphic on a neighborhood of the disk $|z - z_0| \le r$, applying Theorem 12 to $f - a$ we obtain:

**Proposition 4.** *If* $f(z) \neq a$ *on the circle* $|z - z_0| = r$ *then*

$$\frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f'(z)}{f(z)-a}\,dz = \#\{z : f(z) = a, |z - z_0| < r\}, \tag{1.28}$$

*where* $\#$ *means each root of* $f(z) = a$ *is counted as many times as the order of* $f(z) - a$ *indicates.*

**Remark**: *Denote the right hand side of* (1.28) *by* $N(a)$*, we know that* $N(a)$ *is an integer valued holomorphic function on* $\mathbb{C} \setminus \Gamma$*, where* $\Gamma$ *is the image of the circle* $|z - z_0| = r$ *under* $f$*. In particular,* $N(a)$ *is locally a constant. The following theorem on the local correspondence is an immediate consequence of this result.*

**Theorem 14** (Page 131, Theorem 19)**.** *Support that* $f$ *is holomorphic near* $z_0$*,* $f(z_0) = w_0$*, and that* $f(z) - w_0$ *has a zero of order* $n$ *at* $z_0$*. If* $\varepsilon > 0$ *is sufficiently small then*

$$n = \#\{z : f(z) = a, |z - z_0| < \varepsilon\},$$

*for all* $a$ *in some neighborhood of* $w_0$*, moreover, if* $\varepsilon$ *is sufficiently small, then the above equality is also true without counting multiplicity.*

*Proof.* Since the zero of $f(z) - w_0$ is isolated, it suffices to choose $\varepsilon$ so that $f(z)$ is defined and holomorphic on the disk $|z - z_0| \le \varepsilon$ and so that $z_0$ is the only zero of $f(z) - w_0$ in this disk. For the final statement, it suffices to take $\varepsilon$ so that

$$f'(z) \neq 0 \;\text{ for } 0 < |z - z_0| < \varepsilon,$$

then all zeros of $f(z) - a$ are of order one for $a$ in a small punctured neighborhood of $w_0$. □

**Remark**: *By the above theorem, if* $z_0$ *is a finite order zero of* $f(z) - w_0$ *then the* $f$ *image of every sufficiently small disk* $|z - z_0| < \varepsilon$ *contains a neighborhood of* $w_0$*. Hence we have*

**Corollary 4** (Page 132, Corollary 1)**.** *A non-constant holomorphic function maps open sets onto open sets,*

Notice that the followings are equivalent in Theorem 14:

(1) $n = 1$;

(2) $f'(z_0) \neq 0$;

(3) $f$ is one to one near $z_0$.

**Corollary 5** (Page 132, Corollary 2). *If $f$ is holomorphic near $z_0$ then $f'(z_0) \neq 0$ if and only if $f$ maps a neighborhood of $z_0$ one to one onto a neighborhood of $f(z_0)$. The inverse of a one to one holomorphic mapping is also one to one holomorphic. (30th Jan)*

**Remark**: *In fact, there is a very precise formula for the inverse of a one to one holomorphic mapping. To find that formula we need to generalize* (1.28) *to*

$$\frac{zf'(z)}{f(z)-a},$$

*which has residue $z(a)\mathrm{Ord}_{z(a)}(f-a)$ at a zero $z(a)$ of $f(z)-a$. If $f$ is one to one then*

$$z(a)\mathrm{Ord}_{z(a)}(f-a) = z(a) = f^{-1}(a),$$

*which proves the following theorem.*

**Theorem 15.** *Assume that $f$ is one to one holomorphic near $|z-z_0| \leq r$, then*

$$f^{-1}(w) = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{zf'(z)}{f(z)-w}\, dz \tag{1.29}$$

*holds true for $w$ in a neighborhood of $f(z_0)$.*

To count the number of zeros for a general holomorphic function, the following Rouché's theorem (see the corollary in page 153 of the Ahlfors book for generalizations) is often useful.

**Theorem 16** (Rouché's theorem). *Let $f, g$ be holomorphic near $|z-z_0| \leq r$. Assume that*

$$|f-g| < |f| \text{ on } |z-z_0| = r,$$

*then $f, g$ have the same number (counting multiplicity) of zeros in the open disk $|z-z_0| < r$.*

*Proof.* Notice that $F := g/f$ satisfies

$$|1-F| < 1$$

on the circle $|z-z_0| = r$. Denote the $F$ image of the circle $|z-z_0| = r$ by $\Gamma$. We know that $\Gamma$ is a smooth closed curve in the disk $|w-1| < 1$. Thus Theorem 4 gives (note that $1/w$ is holomorphic in the disk $|w-1| < 1$ and the disk is always simply connected)

$$\int_\Gamma \frac{1}{w}\, dw = 0.$$

Replace $w$ by $F(z)$, we get

$$\int_{|z-z_0|=r} \frac{F'(z)}{F(z)}\, dz = 0,$$

thus by Theorem 12, $F$ has the same number (counting multiplicity) of zeros and poles in the open disk $|z-z_0| < r$. Hence the theorem follows. $\square$

## 2. Conformal mapping and the Riemann mapping theorem

### 2.1. The maximum principle, Schwarz Lemma and conformal mapping.

**Theorem 17** (The maximum principle)**.** *If $f$ is holomorphic and non-constant in a domain $\Omega \subset \mathbb{C}$, then its absolute value $|f(z)|$ has no maximum in $\Omega$.*

*Proof.* Use the Cauchy integral formula to prove the submean inequality for $|f|$, which implies that the set $\{|f| = \sup |f|\}$ is open (and closed), thus it is either empty or equal to $\Omega$. This is the second proof given by Ahlfors, page 134-135 (another proof is to use Corollary 4, try!). □

A very useful corollary of the maximum principle is the following:

**Theorem 18** (Lemma of Schwarz)**.** *If $f$ is holomorphic for $|z| < 1$ and satisfies the conditions $|f(z)| \le 1$, $f(0) = 0$, then $|f(z)| \le |z|$ and $|f'(0)| \le 1$. If $|f(z)| = |z|$ for some $z \neq 0$, or if $|f'(0)| = 1$, then $f(z) = cz$ with a constant $c$ of absolute value $1$.*

*Proof.* It suffices to apply the maximum principle to $f(rz)/z$ with $r < 1$ tends to $1$, which implies that $|f(z)/z| \le 1$. Thus $|f(z)| \le |z|$ and $|f'(0)| \le 1$. To prove the second part, it suffices to apply the maximum principle to

$$g(z) := \begin{cases} f'(0) & z = 0 \\ f(z)/z & 0 < |z| < 1. \end{cases}$$

□

*Exercise 1*: This is for the applications of the Schwarz lemma (see [A1, Chapter 1] for more related results).

(1) Fix positive constants $M, R$ and $|w_0| < M$, $|z_0| < R$. Verify that

$$S(w) = \frac{M(w - w_0)}{M^2 - \overline{w_0} w}$$

maps $|w| < M$ one to one onto $|S(w)| < 1$ and

$$T(z) = \frac{R(z - z_0)}{R^2 - \overline{z_0} z}$$

maps $|z| < R$ one to one onto $|T(z)| < 1$.

*Solution*: Note that if $|z| = R$ then

$$|T(z)| = \left|\frac{R(z - z_0)}{z\bar{z} - \overline{z_0} z}\right| = 1.$$

Thus the maximum principle implies that $|T(z)| < 1$ for $|z| < R$. The existence of $T^{-1}$ in (2) implies that $T$ is one to one onto $|T(z)| < 1$.

(2) Show that the inverse of $T$ is given by

$$T^{-1}(\zeta) = \frac{R(R\zeta + z_0)}{R + \overline{z_0}\zeta}.$$

*Solution*: One may directly verify that

$$\frac{R(z-z_0)}{R^2-\overline{z_0}z}=\zeta$$

is equivalent to

$$z=\frac{R(R\zeta+z_0)}{R+\overline{z_0}\zeta}.$$

(3) Assume that $f$ is holomorphic for $|z|<R$ and satisfies the conditions $|f(z)|\leq M$ and $f(z_0)=w_0$. Apply the Schwarz lemma to $S\circ f\circ T^{-1}$ and prove that

$$|\frac{M(f(z)-f(z_0))}{M^2-\overline{f(z_0)}f(z)}|\leq|\frac{R(z-z_0)}{R^2-\overline{z_0}z}|,\ \ \forall\, z,z_0 \text{ with } |z|<R,\ |z_0|<R. \tag{2.1}$$

*Solution*: The Schwarz lemma implies that

$$|S\circ f\circ T^{-1}(\zeta)|\leq|\zeta|$$

for $|\zeta|<1$. Take $\zeta=T(z)$, we obtain

$$|S(f(z))|\leq|T(z)|,$$

which is equivalent to (2.1).

*Exercise 2*: Use (2.1) to prove that: if $f$ is holomorphic for $|z|<R$ and satisfies $|f(z)|\leq M$, then

$$|f'(z_0)|\leq\frac{R(M^2-|f(z_0)|^2)}{M(R^2-|z_0|^2)},\ \ \forall\,|z_0|<R. \tag{2.2}$$

*Solution*: Note that (2.1) gives

$$|\frac{f(z)-f(z_0)}{z-z_0}|\leq|\frac{R(M^2-\overline{f(z_0)}f(z))}{M(R^2-\overline{z_0}z)}|,$$

it suffices to take $z\to z_0$.

*Exercise 3*: Show that a one to one holomorphic mapping $f$ of the unit disk onto itself is given by

$$f(z)=c\frac{z-z_0}{1-\overline{z_0}z} \tag{2.3}$$

for some constant $c$ with $|c|=1$ and $z_0$ with $|z_0|<1$.

*Solution*: Assume that $f$ is one to one holomorphic from the unit disk onto itself and $f(z_0)=0$. Put

$$T(z)=\frac{z-z_0}{1-\overline{z_0}z}.$$

Apply the Schwarz lemma to $f\circ T^{-1}$ and $T^{-1}\circ f$, we obtain $|f|\leq|T|$ and $|T|\leq|f|$. Thus the holomorphic function $f/T$ satisfies that $|f/T|=1$ and it suffices to use Exercise 3 in page 17.

### 2.2. Riemann mapping theorem.

**Definition 11.** *Let $f$ be a holomorphic function on a domain $\Omega_1 \subset \mathbb{C}$. If $f(\Omega_1) \subset \Omega_2$ for some domain $\Omega_2 \subset \mathbb{C}$ then we call*

$$f : \Omega_1 \to \Omega_2$$

*a **holomorphic mapping** from $\Omega_1$ to $\Omega_2$. Assume further that $f$ is one to one and $f(\Omega_1) = \Omega_2$, then we say that $f$ is a **biholomorphic mapping (or conformal mapping)** from $\Omega_1$ to $\Omega_2$.*

**Remark**: *By Corollary 5, we know that if $f$ is conformal then $f^{-1}$ is also conformal.*

**Theorem 19** (Page 230, Theorem 1)**.** *Given any simply connected domain $\Omega \subset \mathbb{C}$ which is not the whole plane, and a point $z_0 \in \Omega$, there exists a unique conformal mapping $f$ from $\Omega$ onto the unit disk $|w| < 1$ normalized by the conditions $f(z_0) = 0$ and $f'(z_0) > 0$.*

*Proof. 1. Uniqueness.* If $f_1$ and $f_2$ are two such mappings, then $S := f_1 f_2^{-1}$ defines a conformal mapping of $|w| < 1$ onto itself with $S(0) = 0$ and $S'(0) > 0$. By Exercise 3 above, we know that $S$ must be the identity mapping. Hence $f_1 = f_2$.

The existence part is divided into the following steps.

*2. Define $f$ as the solution of an optimization problem.* Consider the following

$$\text{optimization problem : Find } f \in \mathcal{F} \text{ with } f'(z_0) = B := \sup\{g'(z_0) : g \in \mathcal{F}\},$$

where

$$\mathcal{F} := \left\{ g : g'(z_0) > 0,\ g(z_0) = 0,\ \sup_{z \in \Omega} |g(z)| \le 1 \text{ and } g : \Omega \to g(\Omega) \text{ is conformal} \right\}.$$

We shall show that the solution $f$ of this optimization problem exists and fits our needs.

*3. $\mathcal{F}$ is not empty.* We note there exists, by assumption, a point $a \notin \Omega$. Since $\Omega$ is simply connected, $h(z) := \sqrt{z-a}$ is well defined in $\Omega$ by Corollary 1. Note that if $h(z_1) = \pm h(z_2)$ then $(h(z_1))^2 = z_1 - a = z_2 - a = (h(z_2))^2$ gives $z_1 = z_2$. Hence we know that $h : \Omega \to h(\Omega)$ is conformal and

$$h(\Omega) \cap -h(\Omega) = \emptyset. \tag{2.4}$$

By Corollary 4, $h(\Omega)$ is open thus covers a disk $|w - h(z_0)| < \rho$, thus (2.4) gives

$$h(\Omega) \cap \{w \in \mathbb{C} : |w + h(z_0)| < \rho\} = \emptyset.$$

In other words, $|h(z) + h(z_0)| \ge \rho$ for all $z \in \Omega$. Then one map verify that the function

$$g_0(z) := \frac{\rho}{4} \frac{|h'(z_0)|}{|h(z_0)|^2} \cdot \frac{h(z_0)}{h'(z_0)} \cdot \frac{h(z) - h(z_0)}{h(z) + h(z_0)}$$

belongs to $\mathcal{F}$ (see page 230 of the Ahlfors book for details). (5th Feb).

*4. Solve the optimization problem.* A priori, the constant $B$ in our optimization problem could be infinite. In any case, one may take $g_n \in \mathcal{F}$ with $g_n'(z_0) \to B$ as $n \to \infty$. By Theorem 23 in section 2.5, $\{g_n\}$ contains a subsequence, say $\{g_{n_k}\}$ which converges locally uniformly to a holomorphic function $f$ on $\Omega$. It is clear that $|f| \le 1$ on $\Omega$, $f(z_0) = 0$ and by Theorem 21 in section 2.4, $f'(z_0) = B$ (this proves that $B < \infty$). Lemma 1 in section 2.4 further implies that $f : \Omega \to f(\Omega)$ is conformal. Hence $f \in \mathcal{F}$ solves the optimization problem.

*5. $f(\Omega)$ is the unit disk.* Otherwise $w_0 \notin f(\Omega)$ for some $|w_0| < 1$. Again since $\Omega$ is simply connected,

$$F(z) = \sqrt{\frac{f(z) - w_0}{1 - \overline{w_0} f(z)}}$$

is well defined in $\Omega$ by Corollary 1. Similar to $h$, we know that $F : \Omega \to F(\Omega)$ is conformal and $|F| \leq 1$. To normalize it we form

$$G(z) := \frac{|F'(z_0)|}{F'(z_0)} \cdot \frac{F(z) - F(z_0)}{1 - \overline{F(z_0)} F(z)},$$

so that $G \in \mathcal{F}$. After brief computation,

$$G'(z_0) = \frac{|F'(z_0)|}{1 - |F(z_0)|^2} = \frac{1 + |w_0|}{2\sqrt{|w_0|}} B > B.$$

This is a contradiction, so $f(\Omega)$ must be the whole unit disk. □

**Remark (Page 231, Ahlfors book).** *At first glance, it may seem like pure luck that our computation yields $G'(z_0) > f'(z_0)$. This is not quite so, for we can write $f = T(G)$ (try to find $T$ yourself) for some holomorphic function $T$ which maps $|w| < 1$ into itself with $T(0) = 0$. The Schwarz lemma gives $|T'(0)| < 1$, thus*

$$|f'(z_0)| = |T'(0) G'(z_0)| < |G'(z_0)|.$$

Another remark is that: $\log |f|$ is usually called the Green function of $\Omega$ with a pole at $z_0$, moreover the constant $B$ in the above proof satisfies

$$B = f'(z_0) = \lim_{z \to z_0} e^{\log(|f(z)|) - \log(|z - z_0|)}.$$

The right hand side is known as the logarithmic capacity of $\mathbb{C} \setminus \Omega$ with respect to $z_0$, which is the central concept in the potential theory (see [Ra, Chapter 5]; in case $\Omega$ is not simply connected, $B$ is also an important conformal invariant, see [AB]). Hence the above proof of the Riemann mapping theorem directly leads us to the potential theory. In fact, the Riemann mapping theorem is equivalent to the existence of Green's function on $\Omega$ (see Theorem 4.4.11 in [Ra] for a nice Green's function proof of the Riemann mapping theorem). In the next few sections, we shall give a complete proof of the results used in Step 4 of the above proof.

### 2.3. Elementary point set topology.

**Definition 12.** *A **topology** on a set $X$ is a collection of subsets of $X$, called **open sets** and satisfying the following conditions:*

1. *The empty set and $X$ itself are open;*
2. *The intersection of any finite number of open sets is open;*
3. *The union of an arbitrary collection of open sets is open.*

*The complement of an open set is called a* ***closed set****. A subset $K$ of $X$ is said to be* ***compact*** *if every open covering of $K$ contains a finite sub-covering.*

**Remark**: *One may verify that open sets in Definition 1 defines a topology on $\mathbb{C}$. The precise meaning of compactness of $K$ is: for every family of open sets $\{U_j\}_{j\in J}$ with*

$$K \subset \bigcup_{j\in J} U_j,$$

*there exists a finite subset, say $\{j_1, \cdots, j_N\}$, of $J$ such that*

$$K \subset U_{j_1} \cup \cdots \cup U_{j_N}.$$

*The following Heine-Borel Theorem gives a precise description of compact sets in $\mathbb{C}$.*

**Theorem 20** (See page 60-61, Ahlfors book for the proof)**.** *A subset in $\mathbb{C}$ is compact if and only if it is closed and bounded.*

We shall mainly use compactness in the following definition:

**Definition 13.** *Let $K$ be a compact subset of an open set $\Omega$ in $\mathbb{C}$. We say that functions $f_n$ on $\Omega$ converges* ***uniformly on $K$ to a function*** *$f$ if*

$$\sup_{z\in K} |f_n(z) - f(z)| \to 0 \text{ as } n \to \infty.$$

*We say that $f_n$ on $\Omega$ converges* ***locally uniformly*** *to a function $f$ on $\Omega$ if for every $z_0 \in \Omega$, there exists an open neighborhood $U$ of $z_0$ such that*

$$\sup_{z\in U} |f_n(z) - f(z)| \to 0 \text{ as } n \to \infty.$$

The definition of compactness immediately gives:

**Proposition 5.** *A sequence of functions converges* ***locally uniformly*** *if and only if it converges* ***uniformly on every compact subset****.*

We shall use this proposition in the next section.

## 2.4. The Weierstrass theorem and the Hurwitz theorem.

**Theorem 21** (The Weierstrass theorem, page 176)**.** *Suppose that a sequence of holomorphic functions $f_n$ converges to $f$ locally uniformly on an open set $\Omega \subset \mathbb{C}$. Then $f$ is holomorphic on $\Omega$; moreover, $f_n'$ converges locally uniformly to $f'$ on $\Omega$.*

*Proof.* For every $a \in \Omega$, take $r$ such that the disk $|z-a| \le r$ lies in $\Omega$. By the Cauchy integral formula, we have

$$f_n(z) = \frac{1}{2\pi i}\int_{|\zeta-a|=r} \frac{f_n(\zeta)\, d\zeta}{\zeta - z},$$

for every $z$ in the open disk $|z-a| < r$. Since the circle $|\zeta - a| = r$ is compact, Proposition 5 implies that $f_n$ converges uniformly to $f$ on that circle. Letting $n \to \infty$ we obtain

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta-a|=r} \frac{f(\zeta)\, d\zeta}{\zeta - z}.$$

Since $f$, as the uniform limit of continuous functions, is always continuous, the above formula implies that $f$ is holomorphic in the open disk $|z-a|<r$. Thus $f$ is holomorphic on $\Omega$. Moreover, starting from

$$f_n'(z)=\frac{1}{2\pi i}\int_{|\zeta-a|=r}\frac{f_n(\zeta)\,d\zeta}{(\zeta-z)^2},$$

the same reasoning yields

$$\lim_{n\to\infty}f_n'(z)=\frac{1}{2\pi i}\int_{|\zeta-a|=r}\frac{f(\zeta)\,d\zeta}{(\zeta-z)^2}=f'(z)$$

and one may verify that the convergence is uniform for $|z-a|<r/2$. □

A non-trivial consequence of the Weierstrass theorem is the following:

**Theorem 22** (The Hurwitz theorem, see page 178, Ahlfors book for the proof)**.** *If the functions $f_n$ are holomorphic and have no zeros on a domain $\Omega\subset\mathbb{C}$, and if $f_n$ converges to $f$ locally uniformly on $\Omega$, then $f$ is either identically zero or never equal to zero in $\Omega$.*

*Proof.* If $f$ is not identically zero then the zeros of $f$ are isolated (of course it includes the case that $f$ has no zeros at all). Hence for every $z_0\in\Omega$, there exists $r>0$ such that $f(z)\neq 0$ on $0<|z-z_0|\leq r$. Since $f_n$ converges locally uniformly to $f$, Theorem 21 and Proposition 5 imply that $f_n$ and $f_n'$ converge uniformly to $f$ and $f'$ on the compact set $|z-z_0|=r$, which gives

$$\lim_{n\to\infty}\frac{1}{2\pi i}\int_{|z-z_0|=r}\frac{f_n'(z)}{f_n(z)}\,dz=\lim_{n\to\infty}\frac{1}{2\pi i}\int_{|z-z_0|=r}\frac{f'(z)}{f(z)}\,dz,$$

by Theorem 12, the integrals in the left hand side are all zero, thus the right hand side is also zero, which gives (by Theorem 12) $f(z_0)\neq 0$. Since $z_0$ is arbitrary, the theorem follows. □

The above theorem can be used to study one to one holomorphic mappings (also called univalent function in page 230 of the Ahlfors book).

**Lemma 1** (The Hurwitz Lemma, page 231)**.** *If a sequence of one to one holomorphic mappings $f_n$ converges to $f$ locally uniformly on a domain $\Omega\subset\mathbb{C}$, then $f$ is either a constant or one to one holomorphic on $\Omega$.*

*Proof.* It suffices to apply the Hurwitz thorem to $f_n(z)-f_n(z_0)$ and the domain $\Omega\setminus\{z_0\}$ (why it is still a domain, try!) for an arbitrarily fixed point $z_0$ in $\Omega$. (6th Feb) □

### 2.5. **Normal families.**

**Definition 14** (Definition 2, page 220)**.** *A family $\mathcal{F}$ of holomorphic functions on a domain $\Omega\subset\mathbb{C}$ is said to be **normal** if every sequence $\{f_n\}$ of functions $f_n\in\mathcal{F}$ contains a subsequence which converges locally uniformly on $\Omega$.*

**Theorem 23** (Montel's theorem)**.** *If $|f|\leq 1$ for all $f\in\mathcal{F}$ then $\mathcal{F}$ is normal.*

*Proof.* Let $\{f_n\}$ be a sequence in $\mathcal{F}$. Denote by $\{w_m\}$ the sequence of rational points in $\Omega$. Since $\{f_n(w_1)\}$ is a bounded sequence in $\mathbb{C}$, the Bolzano-Weierstrass theorem gives a convergent subsequence, say $\{f_{1n}(w_1)\}$. Similarly, $\{f_{1n}(w_2)\}$ has a convergent subsequence, say $\{f_{2n}(w_2)\}$. Continue this process, we obtain a double sequence $\{f_{kn}\}$ with diagonal $f_{nn}$ satisfying

$$\lim_{n\to\infty} f_{nn}(w_k) \text{ exists for all } k \geq 1. \tag{2.5}$$

It suffices to check that $f_{nn}$ converges locally uniformly. Write $B_r(a) := \{z \in \mathbb{C} : |z-a| < r\}$. Assume that $B_{2r}(a) \subset \Omega$, then the Cauchy integral formula gives (see (1.13))

$$\frac{f_{nn}(z) - f_{nn}(w_k)}{z - w_k} = \frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f_{nn}(\zeta)}{(\zeta - w_k)(\zeta - z)}\, d\zeta, \tag{2.6}$$

for all $z, w_k \in B_r(a)$. Hence $|f_{nn}| < 1$ gives

$$\frac{|f_{nn}(z) - f_{nn}(w_k)|}{|z - w_k|} \leq \frac{4}{r}, \ \text{ for all } \ w, w_k \in \overline{B_{r/2}(a)}.$$

Since $\overline{B_{r/2}(a)}$ is compact, for every fixed $\varepsilon > 0$, we can find finite points, say $z_1, \cdots, z_M$, in $B_{r/2}(a)$ such that

$$\overline{B_{r/2}(a)} \subset \cup_{j=1}^{M} B_{\varepsilon r/32}(z_j).$$

Fix a rational $w_j \in B_{\varepsilon r/32}(z_j) \cap B_{r/2}(a)$ for each $1 \leq j \leq M$. Thus for every $z \in B_{r/2}(a)$, there exists $1 \leq k \leq M$ such that $z \in B_{\varepsilon r/32}(z_k)$. Let us apply

$$\begin{aligned} |f_{nn}(z) - f_{mm}(z)| &\leq |f_{nn}(z) - f_{nn}(w_k)| + |f_{nn}(w_k) - f_{mm}(w_k)| + |f_{mm}(w_k) - f_{mm}(z)| \\ &\leq \frac{8|z - w_k|}{r} + |f_{nn}(w_k) - f_{mm}(w_k)| \\ &\leq \frac{8|z - w_k|}{r} + \max_{1\leq j\leq M} |f_{nn}(w_j) - f_{mm}(w_j)| \end{aligned} \tag{2.7}$$

Note that $z, w_k \in B_{\varepsilon r/32}(z_k)$ gives

$$\frac{8|z - w_k|}{r} \leq \frac{\varepsilon}{2}.$$

By (2.5), we can further take $N$ such that

$$\max_{1\leq j\leq M} |f_{nn}(w_j) - f_{mm}(w_j)| \leq \frac{\varepsilon}{2}, \ \text{for all } n, m \geq N.$$

Hence (2.7) gives

$$\sup_{|z-a|<r/2} |f_{nn}(z) - f_{mm}(z)| \leq \varepsilon, \ \text{for all } n, m \geq N.$$

Thus we know that $\{f_{nn}\}$ converges locally uniformly on $\Omega$. (12th Feb) □

**Remark.** *See Theorem 15 in page 224 of the Ahlfors book for an equivalent description of normal families, see also paper 222-224 there for the related Arzela-Ascoli theorem.*

# 3. Harmonic functions

## 3.1. Definitions and basic properties.

**Definition 15.** *A real valued smooth function $u$ on a domain $\Omega \subset \mathbb{C}$ is said to be **harmonic** if*

$$u_{z\bar{z}} = 0, \quad u_{z\bar{z}} := \frac{\partial^2 u}{\partial z \partial \bar{z}}.$$

**Remark.** *If $f = u + iv$ is holomorphic then*

$$0 = f_{z\bar{z}} = u_{z\bar{z}} + i v_{z\bar{z}},$$

*since both $u_{z\bar{z}}$ and $v_{z\bar{z}}$ are real, they must vanish. Hence the real and imaginary parts of a holomorphic function are always harmonic. Later we shall prove a partial converse (see Theorem 24 below): a harmonic function is locally the real part of a holomorphic function.*

**Theorem 24.** *Let $u$ be a harmonic function on a domain $\Omega \subset \mathbb{C}$. If $\Omega$ is simply connected then $u = \operatorname{Re} f$ for some $f$ holomorphic on $\Omega$. Moreover $f$ is unique up to adding a constant.*

*Proof. Uniqueness*: If $u = \operatorname{Re} f$ for some holomorphic function $f$, say $f = u + iv$, then

$$u_x + i v_x = f_x = f' = \frac{f_y}{i} = \frac{u_y + i v_y}{i}$$

gives $u_x = v_y, \ u_y = -v_x$ and

$$df = du + i\, dv = du + i \star du, \tag{3.1}$$

where

$$\star du := -u_y\, dx + u_x\, dy \tag{3.2}$$

is called the conjugate differential of $du$. In particular, (3.1) implies that $f$ is unique up to adding a constant (since $df$ is fully determined by $du$).

*Existence*: (3.1) also suggests to define

$$f(z) := u(z_0) + \int_{z_0}^{z} du + i \star du,$$

where $z_0$ is a fixed point $\Omega$ and the integral is taken over any piecewise smooth curve $\gamma_{z_0,z}$ connecting $z_0, z$ in $\Omega$. Note that (try!)

$$du + i \star du = (u_x - i\, u_y)\, dz \text{ and } u_x - i\, u_y \text{ is holomorphic on } \Omega. \tag{3.3}$$

Since $\Omega$ is simply connected, by the Cauchy integral theorem — Theorem 4, we know that $f(z)$ does not depend on the choice of $\gamma_{z_0,z}$. Hence $f$ is a well defined holomorphic function on $\Omega$ and

$$df = du + i \star du$$

gives $d(\operatorname{Re} f) = du$, together with $\operatorname{Re} f(z_0) = u(z_0)$ we know that $\operatorname{Re} f = u$ on $\Omega$. Hence $f$ fits our needs. Remark: The simply-connected-ness of $\Omega$ is only used in Theorem 4. $\square$

**Remark [page 163, Ahlfors].** *If $\gamma$ is smooth with equation $z = z(t)$, the direction of the tangent is determined by the angle $\alpha = \arg z'(t)$ (i.e. $z'(t) = |z'(t)|e^{i\alpha}$) and we can write*

$$dx = |dz|\,\cos\alpha,\quad dy = |dz|\,\sin\alpha.$$

*The normal which points to the right of the tangent has the direction $\beta = \alpha - \pi/2$, and thus $\cos\alpha = -\sin\beta,\quad \sin\alpha = \cos\beta$. The expression*

$$\frac{\partial u}{\partial n} = u_x \cos\beta + u_y \sin\beta \tag{3.4}$$

*is called the right hand normal derivative of $u$ with respect to the curve $\gamma$. We obtain*

$$\star\, du := -u_y\, dx + u_x\, dy = -u_y|dz|\,\cos\alpha + u_x|dz|\,\sin\alpha = \frac{\partial u}{\partial n}\,|dz| \ \text{ on } \gamma. \tag{3.5}$$

*Thus*

$$\int_\gamma \star\, du = \int_\gamma \frac{\partial u}{\partial n}\,|dz|, \tag{3.6}$$

*see page 164 of the Ahlfors book for more explanations.*

**Theorem 25** (See Ahlfors, page 164, Theorem 19 for another proof)**.** *If $u_1, u_2$ are harmonic on a neighborhood of a piecewise smooth bounded domain $\Omega \subset \mathbb{C}$ then*

$$\int_{\partial\Omega} u_1 \star du_2 - u_2 \star du_1 = 0. \tag{3.7}$$

*Proof.* The idea is to use Green's theorem

$$\int_{\partial\Omega} p\, dx + q\, dy = \int_\Omega (q_x - p_y)\, dxdy. \tag{3.8}$$

We may directly verify (try!) that $q_x - p_y = 0$ in case $p\, dx + q\, dy = u_1 \star du_2 - u_2 \star du_1$. ∎

**Remark.** *For general smooth functions $u, v$, (3.8) gives (try!)*

$$\int_{\partial\Omega} u_1 \star du_2 - u_2 \star du_1 = \int_\Omega (u_1\,\Delta u_2 - u_2\,\Delta u_1)\, dxdy, \tag{3.9}$$

*where the orientation of $\partial\Omega$ is chosen so that $\Omega$ lies to the left and*

$$\Delta u := u_{xx} + u_{yy}$$

*is called the Laplacian of $u$. By (3.5), one may write (3.9) as*

$$\int_{\partial\Omega} \left(u_1\,\frac{\partial u_2}{\partial n} - u_2\,\frac{\partial u_1}{\partial n}\right) |dz| = \int_\Omega (u_1\,\Delta u_2 - u_2\,\Delta u_1)\, dxdy, \tag{3.10}$$

*this is known as Green's formula.* (13th Feb)

*Exercise 1*: (a) With $z = x + iy$, show that for smooth function $u$ we have

$$u_{z\bar{z}} = \frac{1}{4}\left(u_{xx} + u_{yy}\right).$$

Verify that all linear functions $ax + by$ are harmonic.

*Solution*: Use $f_{\bar z} = \frac{1}{2} f_x + \frac{i}{2} f_y$ and $f_{\bar z} = \frac{1}{2} f_x - \frac{i}{2} f_y$.

(b) Check that $\log|z|$ is harmonic on $\mathbb{C}\setminus\{0\}$ and find a holomorphic function $f$ on a simply connected domain $\Omega \subset \mathbb{C}\setminus\{0\}$ such that $\log|z| = \operatorname{Re} f$.

*Solution*: One may just take $f(z) = \log z$ by Corollary 1. $\log|z|$ must be harmonic since it is locally the real part of the holomorphic function $f$.

*Exercise 2*: Use the following steps to show that

$$\star du = r u_r \, d\theta, \ \text{ on } |z| = r, \tag{3.11}$$

for every smooth function $u$ defined on a neighborhood of the circle $|z| = r$.

(a) Check that the normal direction $n$ on $|z| = r$ is given by $n = \frac{(x,y)}{r}$ and $|dz| = r\, d\theta$, then show that

$$\frac{\partial u}{\partial n} |dz| = (x u_x + y u_y)\, d\theta.$$

*Solution*: Clearly we have $n = \frac{(x,y)}{r} = (\cos\theta, \sin\theta)$ and $|dz| = |iz\, d\theta| = r\, d\theta$, thus

$$\frac{\partial u}{\partial n} |dz| = (u_x \cos\theta + u_y \sin\theta)\, r\, d\theta = (x u_x + y u_y)\, d\theta.$$

(b) Write $x = r\cos\theta, \ y = r\sin\theta$, verify that

$$r u_r = x u_x + y u_y,$$

then use $\star du = \frac{\partial u}{\partial n}|dz|$ and (a) to prove (3.11).

*Solution*: It suffices to check that $x u_x + y u_y = r u_r$, which following directly from

$$u_r = u_x \frac{\partial x}{\partial r} + u_y \frac{\partial y}{\partial r} = u_x \cos\theta + u_y \sin\theta = (x u_x + y u_y)/r.$$

*Exercise 3*: Apply (3.7) to the case that

$$\Omega = \{z \in \mathbb{C} : r_1 < |z| < r_2\}, \ u_1(z) = \log|z|, \ u_2 = u,$$

where $u$ is a fixed harmonic function on $s < |z| < \rho$ for some $\rho > r_2, s < r_1$.

(a) Show that (3.7) gives

$$\int_{|z|=r_2} \log|z| \star du - u \star d\log|z| = \int_{|z|=r_1} \log|z| \star du - u \star d\log|z|. \tag{3.12}$$

*Solution*: By (3.7) and $\partial\Omega = \{|z| = r_2\} - \{|z| = r_1\}$.

(b) Apply (3.11) to $\log|z|$, show that $\star\, d\log|z| = d\theta$ on $|z| = r$, then use (3.12) to prove that

$$\int_{|z|=r} (r u_r \log r - u)\ d\theta \text{ does not depend on } s < r < \rho. \tag{3.13}$$

*Solution*: $\star\, d\log|z| = r(\log r)_r\, d\theta = d\theta$, hence (3.12) implies $f(r_1) = f(r_2)$ for $f(r) := \int_{|z|=r} (r u_r \log r - u)\ d\theta$, since $r_1, r_2$ are arbitrary, we know that $f(r)$ is a constant.

(c) Apply (3.7) to the case that $u_1 = 1, u_2 = u$, then show that

$$\int_{|z|=r} r u_r \, d\theta \text{ does not depend on } s < r < \rho. \tag{3.14}$$

*Solution*: Similar to (b).

(d) Use (b) and (c) to prove that: if $u$ is harmonic function on $s < |z| < \rho$, then there exist constants $\alpha, \beta$ such that

$$\frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) \, d\theta = \alpha \log r + \beta$$

for every $s < r < \rho$; moreover we have $\alpha = 0$ if $u$ is harmonic on $|z| < \rho$.

*Solution*: By (b) and (c), we know that

$$c_1 = \int_{|z|=r} (r u_r \log r - u) \, d\theta = c_2 \log r - \int_{|z|=r} u \, d\theta$$

for some constants $c_1$ and $c_r$. Thus the first statement follows. In case $u$ is on $|z| < \rho$ then we have

$$c_2 = \int_{|z|=r} r u_r \, d\theta = \int_{|z|=r} x u_x + y u_y \, d\theta \to 0$$

as $r \to 0$. Thus $\int_{|z|=r} u \, d\theta$ is a constant, another proof is to apply (3.7) to $\Omega = \{|z| < r\}$.

3.2. **The mean-value property.** A nice application of Theorem 24 is the following:

**Theorem 26** (Mean-value property). *Let $u$ be a function harmonic on an open neighborhood of the disk $|z - z_0| \leq r$. Then*

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) \, d\theta. \tag{3.15}$$

*Proof.* By Theorem 24, $u$ is the real part of a holomorphic function $f$, one may check that (try!) the Cauchy integral formula for $f$ gives (3.15). □

(3.15) leads directly to the maximum principle for harmonic functions.

**Theorem 27** (Maximum Principle). *Let $u$ be a harmonic function on a domain $\Omega \subset \mathbb{C}$.*

(1) *If $u$ attains a maximum on $\Omega$ then $u$ is a constant;*
(2) *Assume that $\Omega$ is bounded. If $u$ extends continuously to $\overline{\Omega}$ and $u \leq 0$ on $\partial\Omega$, then $u \leq 0$ on $\Omega$.*

*Proof.* The proof of (1) is the same as for Theorem 17. For the proof of (2), as $\overline{\Omega}$ is compact and $h$ is continuous there, $h$ must attain a maximum at some point $z_0 \in \overline{\Omega}$, i.e.

$$h(z_0) = \sup_{\overline{\Omega}} h.$$

If $z_0 \in \partial\Omega$, then $h(z_0) \leq 0$ by assumption, and so $h \leq 0$ on $\Omega$. If $z_0 \in \Omega$ then (1) implies that $h$ is a constant on $\Omega$. Since $h$ is continuous on $\overline{\Omega}$, we know $h$ must be a constant on $\overline{\Omega}$. Thus in this case, our assumption also implies $h \leq 0$ on $\Omega$. □

**Definition 16** (Maximum principle)**.** *We say that a function $h : \Omega \to [-\infty, \infty)$ satisfies the **maximum principle** on a domain $\Omega \subset \mathbb{C}$ if $h$ has no maximum in $\Omega$ unless it is a constant (i.e. either $h$ is a constant or $h(z) < \sup_\Omega h$, for all $z \in \Omega$).*

**Remark.** *Later we shall study the class of subharmonic functions, which can be defined directly using the maximum principle (see Definition 18).*

3.3. **Poisson's formula.** We shall follow page 166-168 of the Ahlfors book [A0]. The maximum principle, Theorem 27 has the following consequence:

**Corollary 6.** *If $u$ is continuous on a compact set $K$ in $\mathbb{C}$ and harmonic on the interior of $K$, then it is uniquely determined by its value on $\partial K$ (see the proof below for the precise meaning).*

*Proof.* Let $u_1$ and $u_2$ be two such functions with the same boundary values. Apply Theorem 27 (2) to $h = u_1 - u_2$ and $\Omega =$ the interior of $K$, then $\partial\Omega \subset \partial K$ and we obtain $u_1 \leq u_2$ on $K$. Consider $h = u_2 - u_1$ instead, we also get $u_2 \leq u_1$. Thus $u_1 = u_2$ on $K$. □

**Remark.** *The above corollary suggests the following:*

Problem: Find such $u$ with given boundary values.

In this section, we shall solve the problem in case that $K$ is a disk. (3.15) determines the value of $u$ at the center of the disk. But this is all we need, for there exists a linear transformation

$$z = S(\zeta) = \frac{R(R\zeta + a)}{R + \bar{a}\zeta}, \quad |a| < R, \tag{3.16}$$

which maps $|\zeta| \leq 1$ onto $|z| \leq R$ and carries the center $\zeta = 0$ to an arbitrary given point $a$. Suppose that $u(z)$ is harmonic on a neighborhood of $|z| \leq R$, then $u(S(\zeta))$ is harmonic around $|\zeta| < 1$. By (3.15), we obtain

$$u(a) = u(S(0)) = \frac{1}{2\pi} \int_{|\zeta|=1} u(S(\zeta))\, d\arg\zeta. \tag{3.17}$$

From

$$\zeta = \frac{R(z-a)}{R^2 - \bar{a}z},$$

we compute

$$d\arg\zeta = -i\frac{d\zeta}{\zeta} = -i\left(\frac{1}{z-a} + \frac{\bar{a}}{R^2 - \bar{a}z}\right) dz = \left(\frac{z}{z-a} + \frac{\bar{a}z}{R^2 - \bar{a}z}\right) d\theta.$$

Since $|\zeta| = 1$ corresponds to $|z| = R$, i.e $z\bar{z} = R^2$, we have

$$\frac{\bar{a}z}{R^2 - \bar{a}z} = \frac{\bar{a}z}{z\bar{z} - \bar{a}z} = \frac{\bar{a}}{\bar{z} - \bar{a}}.$$

Thus

$$d\arg\zeta = \left(\frac{z}{z-a} + \frac{\bar a}{\bar z - \bar a}\right) d\theta, \tag{3.18}$$

hence (3.17) gives:

**Theorem 28** (Poisson's formula). *If $u(z)$ is harmonic on $|z|<R$, continuous for $|z|\le R$. Then*

$$u(a) = \frac{1}{2\pi}\int_{|z|=R} \frac{R^2-|a|^2}{|z-a|^2} u(z)\,d\theta = \frac{1}{2\pi}\int_{|z|=R}\left(\mathrm{Re}\,\frac{z+a}{z-a}\right) u(z)\,d\theta, \tag{3.19}$$

*for all $|a|<R$. In particular, we have $u=\mathrm{Re}\, f$, where*

$$f(a) := \frac{1}{2\pi}\int_{|z|=R} \frac{z+a}{z-a} u(z)\,d\theta \tag{3.20}$$

*is holomorphic on $|a|<R$.*

*Proof.* Replacing $u$ by $u(rz)$ if necessary, it suffices to prove the case that $u$ is harmonic on a neighborhood of $|z|\le R$. Now (3.17) applies, by (3.18), we know that (3.19) follows from

$$\frac{z}{z-a} + \frac{\bar a}{\bar z-\bar a} = \frac{R^2-|a|^2}{|z-a|^2} = \mathrm{Re}\,\frac{z+a}{z-a}. \tag{3.21}$$

(3.20) follows from $d\theta = \frac{dz}{iz}$ on $|z|=R$. □

**Remark.** *In case $u=1$, (3.19) gives*

$$1 = \frac{1}{2\pi}\int_{|z|=R} \frac{R^2-|a|^2}{|z-a|^2}\,d\theta, \tag{3.22}$$

*hence $\frac{1}{2\pi}\frac{R^2-|a|^2}{|z-a|^2}\,d\theta$ defines a probability measure on the circle $|z|=R$ (19th Feb).*

*Exercise 1*: Let $\Omega$ be a smoothly bounded domain in $\mathbb{C}$. Assume that $z_0\in\Omega$ and there exist a smooth function $G$ on a neighborhood of $\bar\Omega\setminus\{z_0\}$ such that $G=0$ on $\partial\Omega$ and $G(z)-\log|z-z_0|$ extends to a harmonic function in $z\in\Omega$. Apply Green's formula (3.10) to

$$\Omega_\varepsilon := \Omega\setminus\{|z-z_0|\le\varepsilon\},$$

$u_1=G,\ u_2=u$ and let $\varepsilon\to0$. Show that

$$u(z_0) = \frac{1}{2\pi}\int_\Omega G\,\Delta u\,dxdy + \frac{1}{2\pi}\int_{\partial\Omega} u\,\frac{\partial G}{\partial n}\,|dz|, \tag{3.23}$$

for every function $u$ smooth on a neighborhood of $\bar\Omega$.

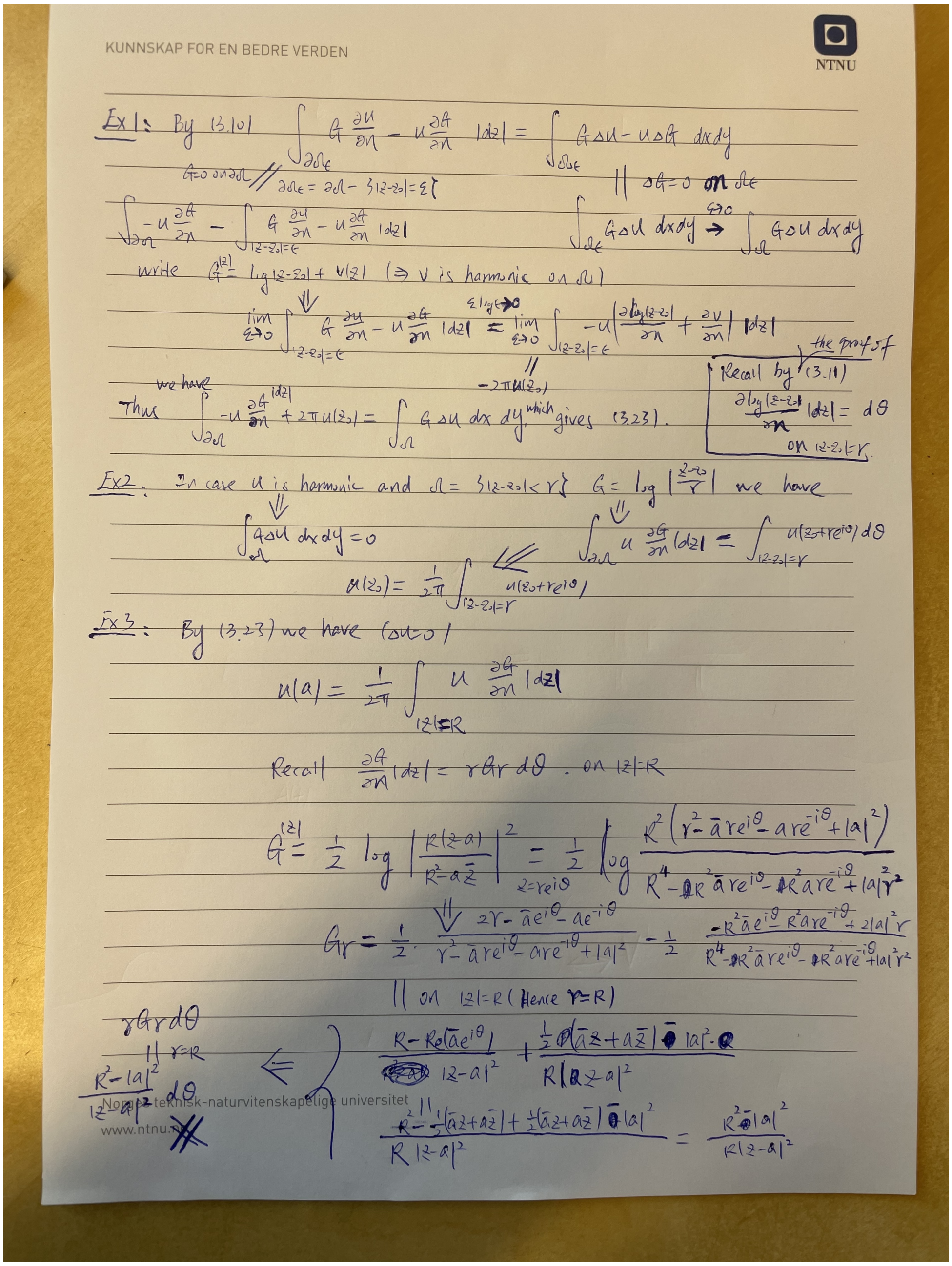
*Exercise 2*: Apply Exercise 1 to $\Omega=\{|z-z_0|<r\}$ and $G=\log|(z-z_0)/r|$, show that (3.23) implies Theorem 26.

*Exercise 3*: Apply Exercise 1 to $\Omega=\{|z|<R\}$ and

$$G(z) = \log\left|\frac{R(z-a)}{R^2-\bar a z}\right|,$$

show that (3.23) implies Theorem 28.

The solution is here:



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**Ex 1:** By (3.10) $\int_{\partial\Omega_\epsilon} G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n}\,|dz| = \int_{\Omega_\epsilon} G\Delta u - u\Delta G\,dxdy$

$G=0$ on $\partial\Omega$ // $\partial\Omega_\epsilon = \partial\Omega - \{|z-z_0|=\epsilon\}$ || $\Delta G = 0$ on $\Omega_\epsilon$

$\int_{\partial\Omega} -u\frac{\partial G}{\partial n} - \int_{|z-z_0|=\epsilon} G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n}\,|dz|$ $\qquad \int_{\Omega_\epsilon} G\Delta u\,dxdy \xrightarrow{\epsilon\to 0} \int_\Omega G\Delta u\,dxdy$

write $G(z) = \log|z-z_0| + v(z)$ ($\Rightarrow v$ is harmonic on $\Omega$)

$$\lim_{\epsilon\to 0}\int_{|z-z_0|=\epsilon} G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n}\,|dz| \overset{\epsilon\log\epsilon\to 0}{=} \lim_{\epsilon\to 0}\int_{|z-z_0|=\epsilon} -u\left(\frac{\partial \log|z-z_0|}{\partial n} + \frac{\partial v}{\partial n}\right)|dz| = -2\pi u(z_0)$$

Recall by the proof of (3.11) $\frac{\partial \log|z-z_0|}{\partial n}|dz| = d\theta$ on $|z-z_0|=r$.

Thus ~~we have~~ $\int_{\partial\Omega} -u\frac{\partial G}{\partial n}|dz| + 2\pi u(z_0) = \int_\Omega G\Delta u\,dx\,dy$, which gives (3.23).

**Ex 2:** In case $u$ is harmonic and $\Omega = \{|z-z_0|<r\}$, $G = \log\left|\frac{z-z_0}{r}\right|$ we have

$\int_\Omega G\Delta u\,dxdy = 0$ $\qquad \int_{\partial\Omega} u\frac{\partial G}{\partial n}|dz| = \int_{|z-z_0|=r} u(z_0+re^{i\theta})\,d\theta$

$$u(z_0) = \frac{1}{2\pi}\int_{|z-z_0|=r} u(z_0+re^{i\theta})$$

**Ex 3:** By (3.23) we have ($\Delta u = 0$)

$$u(a) = \frac{1}{2\pi}\int_{|z|=R} u\frac{\partial G}{\partial n}|dz|$$

Recall $\frac{\partial G}{\partial n}|dz| = rG_r\,d\theta$. on $|z|=R$

$$G(z) = \frac{1}{2}\log\left|\frac{R(z-a)}{R^2-a\bar z}\right|^2 \underset{z=re^{i\theta}}{=} \frac{1}{2}\log\frac{R^2(r^2-\bar a re^{i\theta}-are^{-i\theta}+|a|^2)}{R^4-R^2\bar a re^{i\theta}-R^2are^{-i\theta}+|a|^2r^2}$$

$$G_r = \frac{1}{2}\cdot\frac{2r-\bar a e^{i\theta}-ae^{-i\theta}}{r^2-\bar a re^{i\theta}-are^{-i\theta}+|a|^2} - \frac{1}{2}\,\frac{-R^2\bar a e^{i\theta}-R^2are^{-i\theta}+2|a|^2r}{R^4-R^2\bar a re^{i\theta}-R^2are^{-i\theta}+|a|^2r^2}$$

|| on $|z|=R$ (Hence $r=R$)

$$\frac{R-\mathrm{Re}(\bar a e^{i\theta})}{|z-a|^2} + \frac{\frac{1}{2}(\bar a z+a\bar z) - |a|^2}{R|z-a|^2}$$

$$= \frac{R^2-\frac{1}{2}(\bar a z+a\bar z)+\frac{1}{2}(\bar a z+a\bar z)-|a|^2}{R|z-a|^2} = \frac{R^2-|a|^2}{R|z-a|^2}$$

$rG_r\,d\theta \overset{r=R}{=} \frac{R^2-|a|^2}{|z-a|^2}\,d\theta$

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3.4. **Schwarz's theorem.** Theorem 28 serves to express a given function through its values on a circle. But the right hand side of (3.19) has a meaning as soon as $u$ is defined on $|z| = R$, provided it is sufficiently regular, for instance piecewise continuous. The questions is, does it have the boundary values $u(z)$ on $|z| = R$?

There is reason to clarify the notations. Choosing $R = 1$, we shall introduce the following:

**Definition 17.** *Let $v$ be a piecewise continuous function on $\partial D$, where $D := \{|z - z_0| < r\}$. We call*

$$P_v(z) := \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}\, \frac{e^{i\theta} + \frac{z-z_0}{r}}{e^{i\theta} - \frac{z-z_0}{r}}\, v(z_0 + re^{i\theta})\, d\theta \tag{3.24}$$

*the **Poisson integral of $v$ with respect to the disk $D$**.*

**Remark.** *$P_v$ is linear in $v$:*

$$P_{au+bv} = aP_u + bP_v.$$

*Moreover, $u \geq 0$ implies $P_u(z) \geq 0$; because of this, we call $P_u$ a positive linear functional. We deduce from (3.22) that $P_c = c$ for constant $c$. From this property, we know that any inequality $m \leq u \leq M$ implies $m \leq P_u \leq M$. The question of boundary values is settled by the following fundamental theorem that was first proved by H. A. Schwarz.*

**Theorem 29** (Schwarz's theorem)**.** *The function $P_v$ is harmonic on $D$ and satisfies*

$$\lim_{z \to \zeta} P_v(z) = v(\zeta) \tag{3.25}$$

*provided that $v$ is continuous at $\zeta \in \partial D$.*

*Proof.* Note that $P_v$ is the real part of a holomorphic function, we know that $P_v$ is harmonic on $D$. In proving (3.25) we may assume that $v(\zeta) = 0$, for if this is not the case we only need to replace $v$ by $v - v(\zeta)$. By a change of variable, one may assume that $z_0 = 0$ and $r = 1$ so that $D$ is the unit disk. Write $\zeta = e^{i\theta_0}$, we have

$$P_v(z) = \frac{1}{2\pi} \int_{|\theta-\theta_0|<\varepsilon} \mathrm{Re}\, \frac{e^{i\theta} + z}{e^{i\theta} - z}\, v(e^{i\theta})\, d\theta + \frac{1}{2\pi} \int_{|\theta-\theta_0|\geq\varepsilon} \mathrm{Re}\, \frac{e^{i\theta} + z}{e^{i\theta} - z}\, v(e^{i\theta})\, d\theta$$

By (3.22), the first integral is no bigger than $\sup_{|\theta-\theta_0|<\varepsilon} v(e^{i\theta})$ which tends to $v(e^{i\theta_0}) = v(\zeta) = 0$ as $v$ is assumed to be continuous at $\zeta$. For the second integral, recall that

$$\mathrm{Re}\, \frac{e^{i\theta} + z}{e^{i\theta} - z} = \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \to 0 \text{ uniformly for all } |\theta - \theta_0| > \varepsilon \text{ as } z \to e^{i\theta_0},$$

(note that $|z| \to 1$ and $|e^{i\theta} - z| \to |e^{i\theta} - e^{\theta_0}| > C > 0$ as $|\theta - \theta_0| > \varepsilon$), so the second integral also tends to zero as $z \to e^{i\theta_0}$. It follows that $P_v(z) \to 0 = v(\zeta)$ as $z \to \zeta = e^{i\theta_0}$. □

3.5. **Functions with the mean value property.** A nice application of Schwarz's theorem is the following:

**Theorem 30** (Ahlfors, page 242)**.** *A continuous function $f$ on a domain $U \subset \mathbb{C}$ is harmonic if and only if it satisfies the mean value property*

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta$$

*for all disk $|z - z_0| \leq r$ contained in $\Omega$.*

*Proof.* It suffices to show that $u$ is harmonic on all disk $D$ with $\bar{D} \subset \Omega$. Denote by $v$ the restriction of $u$ to $\partial D$. By Theorem 29, we know that $P_v$ is harmonic on $D$, continuous on $\bar{D}$ and $P_v = v = u$ on $\partial D$. In particular, we know that $P_v$ satisfies the mean value property for all disks contained in $D$. Thus $P_v - u$ satisfies the mean value property for all disks contained in $D$ by our assumption, hence we know that the maximum principle applies to $P_v - u$ and $u - P_v$, which gives that $P_v = u$ on $D$. Thus $u$ is harmonic on $D$. □

3.6. **Harnack's principle.** Another application of the Poisson formula is the following Harnack's inequality for positive harmonic functions.

**Theorem 31** (Harnack's inequality). *If $u$ is non-negative harmonic on the disk $|z| < \rho$ then*

$$\frac{\rho - r}{\rho + r} u(0) \leq u(z) \leq \frac{\rho + r}{\rho - r} u(0) \tag{3.26}$$

*for all $z$ with $|z| = r < \rho$.*

*Proof.* Choose $s$ with $r < s < \rho$. By the Poisson formula (3.19), we have

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{s^2 - r^2}{|se^{i\theta} - z|^2} u(se^{i\theta})\, d\theta.$$

Note that, $s - r \leq |se^{i\theta} - z| \leq s + r$ when $|z| = r$, hence

$$\frac{s-r}{s+r} = \frac{s^2 - r^2}{(s+r)^2} \leq \frac{s^2 - r^2}{|se^{i\theta} - z|^2} \leq \frac{s^2 - r^2}{(s-r)^2} = \frac{s+r}{s-r}$$

gives

$$\frac{s-r}{s+r} \cdot \frac{1}{2\pi} \int_0^{2\pi} u(se^{i\theta})\, d\theta \leq h(re^{i\theta}) \leq \frac{s+r}{s-r} \cdot \frac{1}{2\pi} \int_0^{2\pi} u(se^{i\theta})\, d\theta.$$

By the mean value property, we have $\frac{1}{2\pi} \int_0^{2\pi} u(se^{i\theta})\, d\theta = u(0)$. Letting $s \to \rho$, we know that the above inequality gives (3.26). □

One crucial application of the Harnack inequality is the following Harnack's principle.

**Theorem 32** (Harnack's principle). *Let $u_1 \leq u_2 \leq \cdots$ be harmonic functions on a domain $\Omega \subset \mathbb{C}$. Then either $u_n \to \infty$ locally uniformly or $u_n \to u$ harmonic on $\Omega$ locally uniformly.*

*Proof.* Since $\Omega$ is connected, it suffices to show that the sets on which $\lim u_n(z)$ is, respectively, finite or infinite are both open. In fact, if $u_n(z_0) \to \infty$ as $n \to \infty$ for some $z_0 \in \Omega$, then one may apply the left hand inequality (3.26) to $u_n - u_m, n \geq m$ on the disc $|z - z_0| < \rho$ contained in $\Omega$:

$$\frac{u_n(z_0) - u_m(z_0)}{3} = \frac{\rho - \frac{\rho}{2}}{\rho + \frac{\rho}{2}} (u_n(z_0) - u_m(z_0)) \leq u_n(z) - u_m(z), \quad \text{for all } |z - z_0| \leq \frac{\rho}{2}.$$

It follows that $u_n(z) \to \infty$ uniformly on $|z - z_0| \leq \rho/2$. Similarly, if if $\lim u_n(z_0)$ is finite, then the right hand inequality (3.26) gives, for $n \geq m$

$$0 \leq u_n(z) - u_m(z) \leq 3(u_n(z_0) - u_m(z_0)), \quad \text{for all } |z - z_0| \leq \frac{\rho}{2},$$

which implies that $u_n$ converges uniformly on $|z-z_0| \le \rho/2$. Thus the limit function, say $u$, is also continuous and satisfies the mean value property, hence $u$ is harmonic by Theorem 30. (20th Feb) □

## 4. The Dirichlet problem

*The Dirichlet problem*: Find a harmonic function with given boundary values. In this section, we shall use the Perron family of subharmonic functions to solve the Dirichlet problem and use the reflection principle to show that the solution is harmonic near the boundary if all given data are real analytic.

### 4.1. **Subharmonic functions.**

**Definition 18.** *Let $v$ be a continuous function on a domain $\Omega \subset \mathbb{C}$. $v$ is said to be **subharmonic** if for every $z_0 \in \Omega$, there exists $r_0 > 0$ with $|z-z_0| < r_0$ contained in $\Omega$ such that the following*

$$\text{(4.1)} \qquad \textbf{\textit{submean inequality}}: \quad v(z_0) \le \frac{1}{2\pi}\int_0^{2\pi} v(z_0 + re^{i\theta})\, d\theta$$

*holds for every $0 < r < r_0$.*

**Theorem 33** (Maximum principle)**.** *Let $v$ be subharmonic on a bounded domain $\Omega \subset \mathbb{C}$. If*

$$\limsup_{\Omega \ni z \to \zeta} v(z) \le 0 \ \text{ for all } \ \zeta \in \partial\Omega,$$

*then $v \le 0$ on $\Omega$.*

*Proof.* Since $v$ is bounded near the boundary, we know that

$$M := \sup_{\Omega} v$$

is bounded. The submean inequality implies that $\{v = M\}$ is both closed and open in $\Omega$, hence it is either empty or equal to $\Omega$. If $M > 0$ then our assumption implies that $v(z_0) = M$ for some $z_0 \in \Omega$, hence $\{v = M\} = \Omega$ and $M \le 0$, we get a contradiction. □

**Proposition 6.** *If $v_1, v_2$ are subharmonic, then*

$$c_1 v_1 + c_2 v_2, \quad \max\{v_1, v_2\},$$

*are also subharmonic, where $c_1 \ge 0$, $c_2 \ge 0$ are constants.*

**Proposition 7** (page 247, Ahlfors)**.** *Let $v$ be a subharmonic function on a domain $\Omega \subset \mathbb{C}$. Let $D$ be an open disk with $\overline{D} \subset \Omega$. Then there exists a unique subharmonic function $v_D$ on $\Omega$ such that*

$$\text{(4.2)} \qquad v_D \text{ is harmonic on } D \text{ and } v_D = v \text{ on } \Omega \setminus D.$$

*(We call $v_D$ the **Poisson Modification** of $v$ on $D$).*

*Proof.* By Schwarz's theorem, there is an unique continuous function, say $P_v$, on $\bar{D}$ such that $P_v = v$ on $\partial D$ and $P_v$ is harmonic function on $D$. Hence it suffices to check that

$$v_D = \begin{cases} P_v & \text{on } D \\ v & \text{outside } D \end{cases} \tag{4.3}$$

is subharmonic. Note that $v - v_D = 0$ on $\partial D$, Theorem 33 implies that $v \leq v_D$ on $D$. Hence $v \leq v_D$ on $\Omega$. Now we have

$$v_D(z_0) = v(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} v(z_0 + re^{i\theta})\, d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} v_D(z_0 + re^{i\theta})\, d\theta$$

for every $z_0 \in \partial D$ and small $r > 0$. Since $v_D$ obviously satisfies the submean inequality for small discs outside $\partial D$, we know that $v_D$ is subharmonic everywhere on $\Omega$. □

**Remark**: *Assume that $D$ is given by $|z-a| < r$ in the above theorem, then we have*

$$v(a) \leq v_D(a) = \frac{1}{2\pi}\int_0^{2\pi} v_D(a + re^{i\theta})\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} v(a + re^{i\theta})\, d\theta.$$

*Hence we get:*

**Theorem 34** (Global submean inequality). *Let $v$ be subharmonic on a domain $\Omega \subset \mathbb{C}$. Then*

$$v(a) \leq \frac{1}{2\pi}\int_0^{2\pi} v(a + re^{i\theta})\, d\theta. \tag{4.4}$$

*for every disk $|z-a| \leq r$ in $\Omega$.*

4.1.1. *Test Exam 1.*

*Exercise 1*: Compute the following integrals

$$\int_{|z|=3} \frac{1}{z^2 - 2z}\, dz, \quad \int_{|z|=1} (\operatorname{Im} z)^2\, dz, \quad \int_{-\infty}^{\infty} \frac{dx}{x^4+1}.$$

*Solution*: By the residue theorem, we have

$$\int_{|z|=3} \frac{1}{z^2-2z}\, dz = 2\pi i \left( \operatorname{Res}_2 \frac{1}{z^2-2z} + \operatorname{Res}_0 \frac{1}{z^2-2z} \right) = 2\pi i \left( \frac{1}{2} - \frac{1}{2} \right) = 0.$$

From the definition, we have

$$\int_{|z|=1} (\operatorname{Im} z)^2\, dz = \int_{|z|=1} \frac{z^2 - 2|z|^2 + \bar{z}^2}{-4}\, dz = \int_{|z|=1} \frac{z^2 - 2 + z^{-2}}{-4}\, dz = 0.$$

By the residue theorem, we have

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = 2\pi i \left( \operatorname{Res}_{e^{\pi i/4}} \frac{1}{z^4+1} + \operatorname{Res}_{e^{3\pi i/4}} \frac{1}{z^4+1} \right) = \frac{\pi}{\sqrt{2}}.$$

*Exercise 2*: Show that

$$D := \{z \in \mathbb{C} : |z| < 2\} \setminus \{z \in \mathbb{C} : \operatorname{Re} z = 0,\ |\operatorname{Im} z| < 1\}$$

is not simply connected.

*Solution*: Otherwise, by the Cauchy integral theorem, we should have

$$\int_{|z|=3/2} \frac{1}{z}\, dz = 0,$$

but obviously the above integral is equal to $2\pi i$.

*Exercise 3*: Let $D$ be a bounded domain in $\mathbb{C}$. Let $f$ be a holomorphic function on a neighborhood of $\bar{D}$. Assume that $|f| \le 1$ on $\partial D$. Show that

$$|f'(z_0)| \le \frac{1}{r},$$

for every disk $|z - z_0| \le r$ in $D$.

*Solution*: By the maximum principle, we have $|f| \le 1$ on $D$, thus

$$|f'(z_0)| = \Big|\frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)\, dz}{(z-z_0)^2}\Big| \le \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{r}\, d\theta = \frac{1}{r}.$$

*Exercise 4*: Put $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$. Show that

$$f(z) = \frac{i(z-i)}{z+i}$$

defines a conformal mapping from $\mathbb{H}$ to the unit disk with $f(i) = 0$ and $f'(i) > 0$. Prove further that $f$ is the unique conformal mapping from $\mathbb{H}$ to the unit disk with $f(i) = 0$ and $f'(i) > 0$.

*Solution*: One may verify that the given holomorphic mapping $f$ is one to one and surjective (try to add details yourself!) with $f(i) = 0$ and $f'(i) = \frac{1}{2} > 0$. The second part follows from the uniqueness part of the Riemann mapping theorem. In fact, if $g$ is another function satisfies these properties, then

$$f \circ g^{-1}, \ \ g \circ f^{-1}$$

would be holomorphic mappings from the unit disk to itself, which send the origin back to itself, thus the Schwarz lemma gives

$$|f \circ g^{-1}(z)| \le |z|, \ \ |g \circ f^{-1}(w)| \le |w|.$$

Thus we have $|f| = |g|$. Hence the maximum principle implies that $f/g$ is a constant, i.e. $f = cg$ for some constant $c$ with $|c| = 1$, thus $f'(0), g'(0) > 0$ gives $c = 1$.

*Exercise 5*: Show that $\log|100 + z + z^7|$ is harmonic on a neighborhood of $|z| \le 1$ and then compute

$$\frac{1}{2\pi} \int_0^{2\pi} \log|100 + e^{i\theta} + e^{7i\theta}|\, d\theta.$$

*Solution*: Note that $100+z+z^7$ has no zero on a neighborhood of $|z| \le 1$, thus $\log|100+z+z^7|$ is harmonic there. By the mean value property, we have

$$\frac{1}{2\pi} \int_0^{2\pi} \log|100 + e^{i\theta} + e^{7i\theta}|\, d\theta = \log 100.$$

*Exercise 6*: Put $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Let $f$ be a bounded holomorphic function on a neighborhood of $\overline{\mathbb{H}}$. Use the Cauchy integral formula to show that

$$f(x+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y f(\xi)\, d\xi}{(\xi - x)^2 + y^2}$$

for all $x + iy \in \mathbb{H}$.

*Solution*: By the Residue theorem, we have

$$\int_{-\infty}^{\infty} \frac{f(\xi)\, d\xi}{(\xi - x)^2 + y^2} = 2\pi i \operatorname{Res}_{x+iy} \frac{f(z)}{(z-x-iy)(z-x+iy)} = \frac{\pi f(x+iy)}{y},$$

which gives our formula.

*Exercise 7*: Put $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Let $v$ be a bounded continuous function on $\partial\mathbb{H}$. Show that

$$P_v(x+iy) := \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y v(\xi)\, d\xi}{(\xi - x)^2 + y^2}$$

defines a harmonic function in $x + iy \in \mathbb{H}$ and

$$\lim_{x \to x_0, y \to 0} P_v(x+iy) = v(x_0)$$

for all $x_0 \in \mathbb{R}$.

*Solution*: By the Harnack principle (replace $v$ by $v + C$, it suffices to assume that $v > 0$, then $\{P_v^N\}$ is an increasing family), it suffices to verify that for each $N > 0$,

$$P_v^N(x+iy) := \frac{1}{\pi}\int_{-N}^{N} \frac{y v(\xi)\, d\xi}{(\xi - x)^2 + y^2}$$

is harmonic. Thus, it is enough to show for each $\xi$, $\frac{y}{(\xi-x)^2+y^2}$ is harmonic in $z = x + iy$. Note that

$$\frac{y}{(\xi-x)^2+y^2} = \operatorname{Im} \frac{\xi - x + iy}{(\xi - x + iy)(\xi - x - iy)} = \operatorname{Im} \frac{1}{\xi - z}$$

is the imaginary part of a holomorphic function, thus it must be harmonic. To prove the second part, by Exercise 6, we have $P_c = c$ for constant $c$, thus

$$P_v(x+iy) - v(x_0) = P_{v(x+iy)-v(x_0)} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y(v(\xi) - v(x_0))\, d\xi}{(\xi - x)^2 + y^2}.$$

For each $\varepsilon > 0$, let us choose $\delta_\varepsilon > 0$ such that

$$\sup_{|\xi - x_0| < \delta_\varepsilon} |v(\xi) - v(x_0)| < \varepsilon.$$

Take $M = \sup|2v|$, we know that

$$\left|\int_{-\infty}^{\infty} \frac{y(v(\xi) - v(x_0))\, d\xi}{(\xi - x)^2 + y^2}\right| \leq \int_{|\xi - x_0| < \delta_\varepsilon} \frac{\varepsilon y\, d\xi}{(\xi - x)^2 + y^2} + \int_{|\xi - x_0| \geq \delta_\varepsilon} \frac{M y\, d\xi}{(\xi - x)^2 + y^2}.$$

By Exercise 6, we have

$$\int_{|\xi-x_0|<\delta_\varepsilon} \frac{\varepsilon y\, d\xi}{(\xi-x)^2+y^2} \le \int_{\mathbb{R}} \frac{\varepsilon y\, d\xi}{(\xi-x)^2+y^2} = \pi\varepsilon.$$

By a change of variable $\xi = x+ty$, we also have

$$\int_{|\xi-x_0|\ge\delta_\varepsilon} \frac{My\, d\xi}{(\xi-x)^2+y^2} = \int_{|x-x_0+ty|\ge\delta_\varepsilon} \frac{M\, dt}{t^2+1} \le \int_{|t|\ge\frac{\delta_\varepsilon-(x-x_0)}{y}} \frac{M\, dt}{t^2+1} \to 0$$

as $x\to x_0$ and $y\to 0$. Hence

$$\limsup_{x\to x_0, y\to 0} |P_v(x+iy)-v(x_0)| \le \frac{\pi\varepsilon}{\pi}+0=\varepsilon$$

for every $\varepsilon>0$. Letting $\varepsilon\to 0$, the proof is complete.

*Exercise 8* (Extra): With the notations in Exercise 7, let $v$ be a harmonic function on $\mathbb{H}$. Assume that $v$ is continuous on $\overline{\mathbb{H}}$. Show that if $v$ is bounded then

$$(\star) \qquad v(x+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yv(\xi)\, d\xi}{(\xi-x)^2+y^2}$$

for all $x+iy\in\overline{\mathbb{H}}$ and find a unbounded $v$ such that $(\star)$ does not hold. *Hint: apply the maximum principle to* $v-P_v-\varepsilon\mathrm{Im}\,(\sqrt{iz})$ *and let* $\varepsilon\to 0$.

*Solution*: An example of such unbounded harmonic $v$ is $v(x+iy)=y$. Consider

$$D_R := \{|z|<R, \mathrm{Im}\, z>0\}.$$

By Exercise 7, we know that for each $\varepsilon>0$, $v-P_v-\varepsilon\mathrm{Im}\,(\sqrt{iz})\le 0$ on $\partial D_R$ for all large $R$, thus the maximum principle implies that $v-P_v-\varepsilon\mathrm{Im}\,(\sqrt{iz})\le 0$ on $D_R$ for all large $R$. Hence $v-P_v-\varepsilon\mathrm{Im}\,(\sqrt{iz})\le 0$ on $\mathbb{H}$ for all $\varepsilon>0$. Letting $\varepsilon\to 0$, we get $v\le P_v$. Consider $P_v-v-\varepsilon\mathrm{Im}\,(\sqrt{iz})$ instead, we also get $P_v\le v$. Thus $v=P_v$.

### 4.2. **Solution of the Dirichlet problem.**

**Definition 19** (Dirichlet problem). *Let $\Omega$ be a bounded domain in $\mathbb{C}$. We say that the **Dirichlet problem is solvable** on $\Omega$ if every continuous function on $\partial\Omega$ extends to a continuous function on $\overline{\Omega}$ that is harmonic on $\Omega$.*

**Definition 20** (Perron envelope). *Let $\Omega$ be a bounded domain in $\mathbb{C}$ and let $\phi$ be a continuous function on $\partial\Omega$. We call*

$$u_\phi := \sup\{v : v \text{ is subharmonic on } \Omega \text{ with } v^*\le\phi \text{ on } \partial\Omega\}$$

*the **Perron envelope** of $\phi$ on $\Omega$, where $v^*$ is the function on the closure $\overline{\Omega}$ defined by*

$$v^*(\zeta) = \limsup_{\Omega\ni z\to\zeta} v(z), \quad \zeta\in\overline{\Omega}.$$

**Theorem 35.** *$u_\phi$ is harmonic on $\Omega$.*

*Proof. Step 1: $u_\phi$ is bounded.* Denoe by $\mathcal{B}_\phi$ the collection of all subharmonic $v$ with $v^* \leq \phi$ on $\partial\Omega$. Since the constant function $v = m := \min_{\partial\Omega} \phi$ lies in $\mathcal{B}_\phi$ we know that $u_\phi \geq m$. On the other hand, Theorem 33 gives

$$v \leq M := \max_{\partial\Omega} \phi$$

on $\Omega$ for all $v \in \mathcal{B}_\phi$. Hence we know that $m \leq u_\phi \leq M$ is bounded.

*Step 2: Use the Poisson modification and the Harnack principle.* Consider an open disk $D$ with $\bar{D} \subset \Omega$. Fix $z_0 \in D$. Then one may take a sequence of function $v_n \in \mathcal{B}_\phi$ such that

$$\lim_{n\to\infty} v_n(z_0) = u_\phi(z_0).$$

Set

$$V^n = \max\{v_1, v_2, \cdots, v_n\}.$$

then $V^n$ form a non-decreasing sequence in $\mathcal{B}_\phi$. By Proposition 7, we know that the Poisson modifications $V^n_D$ also lie in $\mathcal{B}_\phi$ and form a non-decreasing sequence. Note that

$$v_n(z_0) \leq V^n(z_0) \leq V^n_D(z_0) \leq u_\phi(z_0)$$

implies that $\lim_{n\to\infty} V^n_D(z_0) = u_\phi(z_0)$. Since $u_\phi(z_0)$ is finite (by Step 1), the Harnack principle implies that $V^n_D$ converges to a harmonic function $U$ on $D$ satisfying $U \leq u_\phi$ and $U(z_0) = u_\phi(z_0)$.

*Step 3: $U = u_\phi$ on $D$.* Let us do Step 2 for another point $z_1 \in \Omega$ and select $w_n \in \mathcal{B}_\phi$ with $\lim_{n\to\infty} w_n(z_1) = u_\phi(z_1)$. But this time, we set

$$W^n := \max\{w_1, v_1, \cdots, w_n, v_n\}.$$

Then $W^n_D$ converge to $U_1$, harmonic on $D$, with

$$U \leq U_1 \leq u_\phi, \quad U_1(z_1) = u_\phi(z_1).$$

Hence we know that $U - U_1$ has the maximum zero at $z_0$. Therefore $U = U_1$ on $D$ by the maximum principle. Thus $u_\phi(z_1) = U(z_1)$ for arbitrary $z_1 \in D$. It follow that $u_\phi = U$ on $D$ and $u_\phi$ is harmonic on any disk $D$ and, consequently, on all of $\Omega$. (26th Feb) $\square$

**Definition 21.** *Let $\Omega$ be a bounded domain in $\mathbb{C}$ and let $\zeta_0 \in \partial\Omega$. A **barrier** at $\zeta_0$ is a continuous function $\omega$ on $\overline{\Omega}$, harmonic on $\Omega$, such that*

$$\omega(\zeta_0) = 0 \text{ and } \omega > 0 \text{ on } \overline{\Omega} \setminus \{\zeta_0\}.$$

*$\Omega$ is said to be **regular** if every boundary point of $\Omega$ possesses a barrier.*

**Theorem 36** (See Ahlfors, page 250)**.** *Let $\Omega$ be a bounded domain in $\mathbb{C}$. The Dirichlet problem is solvable for $\Omega \iff \Omega$ is regular.*

*Proof. Proof of* $\Rightarrow$. If the Dirichlet problem is solvable for $\Omega$ then for every $\zeta_0 \in \partial\Omega$, the function $\phi$ defined by

$$\phi(\zeta) := |\zeta - \zeta_0|^2, \quad \zeta \in \partial\Omega,$$

extends to a continuous function, say $\omega$ on $\bar{\Omega}$, harmonic on $\Omega$. The maximum principle, Theorem 27, for $-\omega$ implies that $\omega$ is positive on $\Omega$. Thus $\omega$ is a barrier at $\zeta_0$. Since $\zeta_0$ is arbitrary, we know that $\Omega$ is regular.

*Proof of* $\Leftarrow$. Assume that $\Omega$ is regular, it suffices to show that for every continuous function $\phi$ on $\partial\Omega$, the Perron envelope $u_\phi$ satisfies that

$$\lim_{z\to\zeta_0} u_\phi(z) = \phi(\zeta_0), \tag{4.5}$$

for every $\zeta_0 \in \partial\Omega$. Since $\Omega$ is bounded, we know that $\phi$ is bounded, so we can take $M > 0$ such that $|\phi| \le M$ on $\partial\Omega$. For every $\varepsilon > 0$, there exists a small disk $D$ around $\zeta_0$ such that

$$|\phi(\zeta) - \phi(\zeta_0)| < \varepsilon,$$

for $\zeta \in D \cap \bar{\Omega}$. Let $\omega$ be a barrier at $\zeta_0$, we have

$$\omega_0 := \inf_{\bar{\Omega}\setminus D} \omega > 0.$$

Consider

$$W(z) := \phi(\zeta_0) + \varepsilon + \frac{\omega(z)}{\omega_0}(M - \phi(\zeta_0)).$$

For $\zeta \in D \cap \partial\Omega$, we have $W(\zeta) \ge \phi(\zeta_0) + \varepsilon > \phi(\zeta)$; for $\zeta \in \partial\Omega \setminus D$ we obtain

$$W(\zeta) \ge \phi(\zeta_0) + \varepsilon + M - \phi(\zeta_0) = M + \varepsilon > \phi(\zeta).$$

By the maximum principle any function $v \in \mathcal{B}_\phi$ must hence satisfy $v < W$. Hence $u_\phi \le W$ and we have

$$\limsup_{z\to\zeta_0} u_\phi(z) \le W(\zeta_0) = \phi(\zeta_0) + \varepsilon. \tag{4.6}$$

For the lower limit, we consider

$$V(z) := \phi(\zeta_0) - \varepsilon - \frac{\omega(z)}{\omega_0}(M + \phi(\zeta_0)).$$

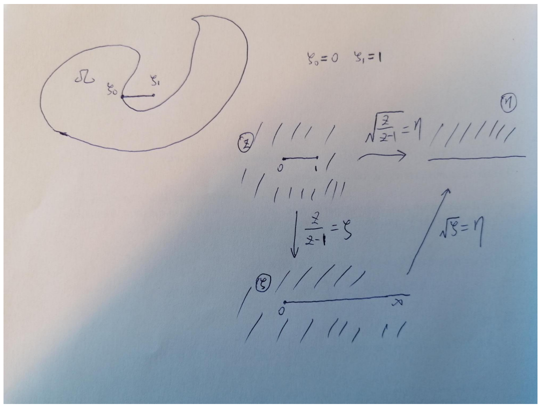
One may verify that $V \in \mathcal{B}_\phi$, hence $u_\phi \ge V$ gives

$$\liminf_{z\to\zeta_0} u_\phi(z) \ge V(\zeta_0) = \phi(\zeta_0) - \varepsilon. \tag{4.7}$$

Since $\varepsilon$ is arbitrary, (4.6) and (4.7) together give (4.5). $\square$

It remains to formulate geometry conditions which imply the existence of a barrier. To begin with the simplest case, suppose that $\bar{\Omega}$ is contained in the half space $\operatorname{Im} z > 0$, except for the $\zeta_0 = 0$ which lies in $\partial\Omega$. Then $w(z) := \operatorname{Im} z$ is a barrier at $\zeta_0$. More generally, suppose that $\zeta_0$ is the end point of a line segment, say $[\zeta_0, \zeta_1]$, all of whose point, except $\zeta_0$, lie in $\mathbb{C} \setminus \bar{\Omega}$. By a linear change of coordinate, let us assume that $\zeta_0 = 0, \zeta_1 = 1$, we know that there is a conformal mapping (see the picture below (should be $\frac{z}{1-z} = \zeta$))

$$\eta(z) := \sqrt{\frac{z}{1-z}},$$



which maps $\mathbb{C} \setminus [0,1]$ onto $\{\operatorname{Im} \eta > 0\} \setminus \{i\}$. Hence

$$\omega(z) := \operatorname{Im} \eta(z) = \operatorname{Im} \sqrt{\frac{z}{1-z}}$$

defines a barrier at $\zeta_0$. To summarize, we have:

**Theorem 37.** *The Dirichlet problem can be solved for any bounded domain $\Omega \subset \mathbb{C}$ such that each boundary point is the end point of a line segment whose other points lie in $\mathbb{C} \setminus \bar{\Omega}$. In particular, any bounded domain with continuous boundary (i.e. the boundary is locally the graph of a continuous function) is regular.*

For simply connected domains, we have the following result (see Theorem 4.2.1 in [Ra] for the proof).

**Theorem 38.** *Every simply connected bounded domain $\Omega \subset \mathbb{C}$ is regular. (27th Feb)*

4.3. **The reflection principle.** The proof of Theorem 30 implies that (try!)

($\star$) *A continue function is harmonic if and only if it satisfies the mean value property locally.*

This fact implies the following reflection principle of Schwarz.

**Theorem 39** (Reflection principle). *Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and write*

$$\mathbb{D}^+ := \{z \in \mathbb{D} : \operatorname{Im} z > 0\}, \quad \sigma := \{z \in \mathbb{D} : \operatorname{Im} z = 0\}.$$

*Suppose that $v$ is continuous on $\mathbb{D}^+ \cup \sigma$, harmonic on $\mathbb{D}^+$ and $v = 0$ on $\sigma$. Then $v$ extends to a harmonic function (still denoted by $v$) on $\mathbb{D}$ satisfying $v(\bar{z}) = -v(z)$ for $z \in \mathbb{D}$.*

*Proof.* One may directly the following continuous function

$$\tilde{v}(z) := \begin{cases} v(z) & z \in \mathbb{D}^+ \\ 0 & z \in \sigma \\ -v(\bar{z}) & \bar{z} \in \mathbb{D}^+ \end{cases}$$

satisfies the mean value property locally, thus ($\star$) above implies that $\tilde{v}$ is harmonic. $\square$

4.4. **Use of the reflection principle.** The reflection principle can be used to study regularity property of the solution of the Dirichlet problem:

$$\Delta u = 0 \text{ on } \Omega \text{ with } u|_{\partial\Omega} = \phi \ \ (u|_{\partial\Omega} \text{ means the restriction of } u \text{ to } \partial\Omega). \tag{4.8}$$

We say that $u$ is a solution of (4.8), if $u$ is harmonic on $\Omega$, continuous on $\overline{\Omega}$ with $u|_{\partial\Omega} = \phi$. Theorem 27 implies that the solution, if it exists, must be unique. The following theorem is a deep result in PDE theory:

**Theorem 40** (See Theorem 9.9 in [A]). *If $\Omega$ is smoothly bounded and $\phi$ is smooth then the solution of (4.8) is unique and smooth up to the boundary (i.e. it extends to a smooth function on a neighborhood of $\overline{\Omega}$).*

We are not able to prove the above result using theories covered in this course. But in case $\Omega$ and $\phi$ are real analytic, we shall show that the reflection principle gives (in fact, a stronger version of) the above result.

**Definition 22.** *We say that $\Omega$ and $\phi$ are **real analytic** if for every $\zeta \in \partial\Omega$, there exists a conformal mapping $f$ from $\mathbb{D}$ onto an open neighborhood, say $V_\zeta$, of $\zeta$ such that*

$$f(\mathbb{D}^+) = \Omega \cap V_\zeta, \quad f(\sigma) = \partial\Omega \cap V_\zeta,$$

*and $\phi(f) = \mathrm{Re}\, h$ on $\sigma$ for some holomorphic function $h$ on $\mathbb{D}$.*

**Theorem 41.** *Assume that $u$ is a solution of (4.8). If $\Omega$ and $\phi$ are real analytic then $u$ extends to a harmonic function on a neighborhood of $\overline{\Omega}$.*

*Proof.* By Definition 22, we know that $\phi(f) = \mathrm{Re}\, h$, hence the harmonic function $u(f) - \mathrm{Re}\, h$ on $\mathbb{D}^+$ vanishes on $\sigma$. By Theorem 39, $u(f) - \mathrm{Re}\, h$ extends to a harmonic function on $\mathbb{D}$. Thus $u$ extends to a harmonic function on $V_\zeta$. Since harmonic functions are real analytic, we know that the extensions to overlapping $V_\zeta$, $\zeta \in \partial\Omega$, must coincide and define a harmonic function on a neighborhood of $\overline{\Omega}$. □

*Exercise 1*: Show that the punctured disk $0 < |z| < 1$ is not regular. *Hint: apply Exercise 3 (d) in page 31.*

*Solution*: If it is regular then it has a barrier, say $\omega$, at $0$. We know that $\omega$ is harmonic on $0 < |z| < 1$, continuous on $|z| \leq 1$, $\omega(0) = 0$ and $\omega(z) > 0$ for all $0 < |z| \leq 1$. By Exercise 3 (d) in page 31, we know that

$$\frac{1}{2\pi}\int_0^{2\pi} \omega(re^{i\theta})\, d\theta = \alpha \log r + \beta, \quad \forall\, 0 < r < 1.$$

Since $\omega$ is bounded near $0$, we know that $\alpha = 0$ and $\beta = \omega(0)$ (try!). Thus the proof of Theorem 30 implies that $\omega$ is harmonic on $|z| < 1$ (try!). Note that $0$ is the maximum point of the harmonic function $-\omega$, by the maximum principle, we know that $\omega = 0$ everywhere on $|z| \leq 1$, which contradicts with the fact that $\omega(z) > 0$ for $0 < |z| \leq 1$.

*Exercise 2*: Prove the following result by reading page 173 in [A0].

**Corollary 7.** *With the notation in Theorem 39. Let $v$ be the imaginary (resp. real) part of an holomorphic function $f$ on $\mathbb{D}^+$. Assume that*

$$\lim_{z\to z_0} v(z) = 0 \text{ for all } z_0 \in \sigma.$$

*Then $f$ extends to a holomorphic function (still denoted by $f$) on $\mathbb{D}$ satisfying $f(z) = \overline{f(\bar z)}$ (resp. $f(z) = -\overline{f(\bar z)}$) for $z \in \mathbb{D}$.*

*Proof.* We only prove the $v = \operatorname{Im} f$ case and leave the $v = \operatorname{Re} f$ case to the readers. By the reflection principle, we know that $v$ extends to a harmonic function $\tilde v$ such that $\tilde v(\bar z) = -\tilde v(z)$ for $z \in \mathbb{D}$. Since $\mathbb{D}$ is simply connected, we know that $\tilde v = \operatorname{Im} \tilde f$ for some holomorphic function $\tilde f$ on $\mathbb{D}$. Now $\tilde f$ and $f$ has the same imaginary part, hence we know that (try!) $\tilde f - f$ is equal to a real constant, say $c$ on $\mathbb{D}^+$. Hence $F := \tilde f - c$ is a holomorphic extension of $f$. Since $\tilde v(\bar z) = -\tilde v(z)$, we know that the holomorphic function $G(z) := \overline{F(\bar z)}$ has the same imaginary part as $F$ and they are equal on $\sigma$, thus $F(z) = G(z) = \overline{F(\bar z)}$. □

*Exercise 3*: Use Corollary 7 to prove the following result (see Theorem 3, 4 in [A0] for related results).

**Theorem 42.** *Let $\Omega$ be a bounded simply connected domain with real analytic boundary in $\mathbb{C}$. Then the Riemann mapping function $f$ which maps $\Omega$ onto the unit disk extends to a holomorphic function on a neighborhood of $\overline{\Omega}$.*

*Proof.* Note that $|f(z)| \to 1$ when $z \to \partial\Omega$, hence $\log f$, which is well defined on $\Omega \cap V_\zeta$ (choose a smaller $V_\zeta$ if necessary, one may assume that $f$ has no zero in $\Omega \cap V_\zeta$), satisfies that

$$\operatorname{Re} \log f(z) = \log |f(z)| \to 0, \ \text{ as } z \to \partial\Omega \cap V_\zeta .$$

Thus by Corollary 7, $\log f$ extends holomorphically to $V_\zeta$. Thus $f$ extends to $V_\zeta$ (hence to $\overline{\Omega}$). □

## 5. Potential theory in the complex plane

### 5.1. Green's functions as envelopes.

5.1.1. *Green's functions for regular bounded domains.* Let $\Omega$ be a bounded domain in $\mathbb{C}$. Fix $w \in \Omega$, assume that $\Omega$ is regular, then by Theorem 36, there exists a

(5.1) harmonic function $u(z)$ on $\Omega$, continuous on $\bar\Omega$ and $u(\zeta) = -\log|\zeta - w|$ for $z \in \partial\Omega$.

**Definition 23** (Definition of Green's function for regular bounded domains)**.** *We call*

$$G_\Omega(z, w) := u(z) + \log|z - w|, \quad z \in \bar\Omega, \ w \in \Omega,$$

*the* ***Green function with a pole at*** *$w \in \Omega$.*

The maximum principle implies the following result:

**Proposition 8.** *Let $\Omega$ be a bounded regular domain in $\mathbb{C}$. Then $z \mapsto G_\Omega(z, w)$ is the unique function on $\Omega$ such that $G_\Omega(z, w) = 0$ for $z \in \partial\Omega$, $G_\Omega(z, w) - \log|z - w|$ (as a function of $z$) is harmonic on $\Omega$ and continuous on $\bar\Omega$.*

5.1.2. *Green's function as an envelope.* By the proof of Theorem 36, we know that the function $u$ in (5.1) satisfies that

$$u = \sup\{v : v \text{ is subharmonic on } \Omega \text{ with } v^*(\zeta) \leq -\log|\zeta - w| \text{ for } \zeta \in \partial\Omega\}$$

hence we get the following result.

**Proposition 9.** *Let $\Omega$ be a bounded regular domain in $\mathbb{C}$. Then*

$$G_\Omega(\cdot, w) = \sup\{\psi \leq 0 : \psi(z) - \log|z - w| \text{ is subharmonic for } z \in \Omega\}, \tag{5.2}$$

*for every fixed $w \in \Omega$.*

Note that the envelope (5.2) is also well defined (and harmonic on $\Omega$ by Theorem 36) for non-regular $\Omega$. Hence, one can use it to define Green's function for general (bounded) domains.

**Definition 24** (Definition of Green's function for general domains, see Definition 10.1 in page 141 in [A1] for the background)**.** *Let $\Omega$ be a domain in $\mathbb{C}$. Fix $w \in \Omega$, we call $\Omega$ a **hyperbolic** domain if there exists $\psi \leq 0$ on $\Omega$ such that $\psi(z) - \log|z - w|$ is subharmonic for $z \in \Omega$. If no such $\psi$ exists, we say that $\Omega$ is **parabolic**. For hyperbolic $\Omega$, we define its Green's function (with a pole at $w \in \Omega$) as*

$$G_\Omega(\cdot, w) = \sup\{\psi \leq 0 : \psi(z) - \log|z - w| \text{ is subharmonic for } z \in \Omega\}.$$

*In case $\Omega$ is parabolic, we define $G_\Omega(\cdot, w) \equiv -\infty$. (4th Mar)*

**Remark.** *The definition of hyperbolicity and parabolicity does not depend on the choice of $w \in \Omega$. In fact, for $w_1, w_2 \in \Omega$, if there exists $\psi_1 \leq 0$ on $\Omega$ such that $\psi_1(z) - \log|z - w_1|$ is subharmonic for $z \in \Omega$.* Assume that Then one may check that

$$C := \sup_{z \in \Omega}\{\psi_1(z) - \log|z - w_1| + \log|z - w_2|\} < \infty.$$

In fact, assume that $w_2 \neq w_1$, then both $\psi_1(z) - \log|z - w_1|$ and $\log|z - w_2|$ are continuous (thus bounded) near $w_1$. But outside a small neighborhood of $w_1$, say for $|z - w_1| \geq \varepsilon$, we must have

$$\psi_1(z) - \log|z - w_1| + \log|z - w_2| \leq \log|\frac{z - w_2}{z - w_1}| \leq |\frac{w_1 - w_2}{z - w_1}| \leq \frac{|w_1 - w_2|}{\varepsilon}.$$

Thus $C$ must be bounded and it suffices to take

$$\psi_2(z) := \psi_1(z) - \log|z - w_1| + \log|z - w_2| - C.$$

We know that $\psi_2 \leq 0$ and $\psi_2(z) - \log|z - w_2|$ is subharmonic.

One may easily verify that every bounded domain is hyperbolic. The Exercise 1 in the end of section 5.3 implies that $\mathbb{C}$ is parabolic. We shall show that the Riemann mapping theorem Theorem (19) implies:

**Proposition 10.** *Let $\Omega$ be a simply connected domain in $\mathbb{C}$. Assume that $\Omega \neq \mathbb{C}$, then $\Omega$ is hyperbolic and $G_\Omega(z, w) = \log|f_w(z)|$, where $f_w$ is the Riemann mapping from $\Omega$ to the unit disk such that $f_w(w) = 0$ and $f'_w(w) > 0$.*

*Proof.* The existence of $f_w$ is just the Riemann mapping theorem. Put

$$\psi := \log |f_w(z)|,$$

we know that $\psi < 0$ and

$$\psi(z) - \log |z-w| = \log \left|\frac{f_w(z)-f_w(w)}{z-w}\right|$$

is harmonic in $z \in \Omega$. Thus we know that $\Omega$ is hyperbolic and $G_\Omega(z,w) \geq \log |f_w(z)|$. The fact that $G_\Omega(z,w) \leq \log |f_w(z)|$ follows from the maximal principle (try!). □

This proposition suggests a *Green function proof* of the Riemann mapping theorem, see the proof of Theorem 4.4.11 in [Ra] for details. In case $\Omega$ is the unit disk $\mathbb{D}$, we know that (try, use (2.3))

$$f_w(z) = \frac{z-w}{1-\bar{w}z},$$

hence we get

$$G_{\mathbb{D}}(z,w) = \log \left|\frac{z-w}{1-\bar{w}z}\right|. \tag{5.3}$$

5.2. **Poisson kernels and harmonic measures.** Assume that $\Omega$ has real analytic boundary, then the reflection principle (see Theorem 41) implies that the Green function $G_\Omega(\cdot, w)$ extends to a harmonic function on a neighborhood of $\bar{\Omega} \setminus \{w\}$. Thus similar to the proof of (5.6), Green's formula applies and implies the following theorem.

**Theorem 43** (See Theorem 4.5.1 in [Ra] for generalizations)**.** *Let $\Omega$ be a bounded domain in $\mathbb{C}$ with real analytic boundary. Then*

$$u(w) = \frac{1}{2\pi}\int_\Omega G_\Omega(z,w)\Delta u(z)\, dxdy + \frac{1}{2\pi}\int_{\partial\Omega} \frac{\partial G_\Omega(z,w)}{\partial n_z} u(z)\,|dz|, \quad w \in \Omega, \tag{5.4}$$

*for all function $u$ smooth on a neighborhood of $\bar{\Omega}$, where $n_z$ denotes the* ***outward unit normal vector*** *at $z \in \partial\Omega$; in particular, if $u$ is harmonic on $\Omega$ then*

$$u(w) = \frac{1}{2\pi}\int_{\partial\Omega} \frac{\partial G_\Omega(z,w)}{\partial n_z} u(z)\,|dz|, \quad w \in \Omega. \tag{5.5}$$

*Proof.* Put $\Omega_\varepsilon := \Omega \setminus \{|z-w| \leq \varepsilon\}$. Since $G_\Omega(z,w)$ is harmonic for $z \in \Omega_\varepsilon$, the Green's formula (3.10) gives

$$\int_{\partial\Omega_\varepsilon} \left( G_\Omega(z,w)\frac{\partial u}{\partial n_z} - u\frac{\partial G_\Omega(z,w)}{\partial n_z} \right) |dz| = \int_{\Omega_\varepsilon} G_\Omega(z,w)\Delta u(z)\, dxdy, \tag{5.6}$$

Since $G_\Omega(z,w) = 0$ for $z \in \partial\Omega$ and $\partial\Omega_\varepsilon = \partial\Omega - \{|z-w| = \varepsilon\}$, we have (see the proof of (3.23) for details)

$$\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon} \left( G_\Omega(z,w)\frac{\partial u}{\partial n} - u\frac{\partial G_\Omega(z,w)}{\partial n} \right) |dz| = 2\pi u(w) - \int_{\partial\Omega} \frac{\partial G_\Omega(z,w)}{\partial n_z} u(z)\,|dz|. \tag{5.7}$$

Hence (5.4) follows by letting $\varepsilon \to 0$. □

**Remark.** *By the maximum principle, if $u \geq 0$ on $\partial\Omega$ and harmonic on $\Omega$ then $u \geq 0$ on $\Omega$, thus* (5.5) *implies that $p(z,w)|dz|$, with*

$$p(z,w) := \frac{1}{2\pi}\frac{\partial G_\Omega(z,w)}{\partial n_z}, \tag{5.8}$$

*defines a probability measure on $\partial\Omega$. Since its integral* (5.5) *gives the value of harmonic functions, we call $p(z,w)|dz|$ the harmonic measure on $\partial\Omega$ with respect to $w \in \Omega$.*

**Definition 25.** *Let $\Omega$ be a bounded domain in $\mathbb{C}$ with real analytic boundary. We call $p(z,w)$, $z \in \partial\Omega$, $w \in \Omega$, defined in* (5.8) *the* ***Poisson kernel*** *of $\Omega$. The corresponding probability measure $p(z,w)|dz|$ is called the* ***harmonic measure*** *on $\partial\Omega$ with respect to $w \in \Omega$. (5th Mar)*

5.3. **Equilibrium measure and logarithmic capacity.** The complement of $1 \leq |z| \leq 2$ in $\mathbb{C}$ contains two parts, $|z| > 2$ and $|z| < 1$; both are connected open sets, we call them *connected components* of $\mathbb{C} \setminus \{1 \leq |z| \leq 2\}$. In general, let $K$ be a compact set in $\mathbb{C}$, we can write

$$\mathbb{C} \setminus K = U_0 \cup U_1 \cup \cdots$$

as a *disjoint union*, each domain $U_j$ is called a connected component. We shall write the unique unbounded one, say $U_0$, as $\Omega_K$. We call $\Omega_K$ the unbounded connected component of $\mathbb{C} \setminus K$.

**Proposition 11.** *Let $K$ be a compact set in $\mathbb{C}$. Denote by $\Omega_K$ the* unbounded connected component *of $\mathbb{C} \setminus K$. Assume that $\Omega_K$ has real analytic boundary. Then there is a unique function $G_{\Omega_K}$ on $\Omega_K$ such that*

*(1) $G_{\Omega_K} \leq 0$ on $\Omega$;*

*(2) $G_{\Omega_K}$ extends to a harmonic function near $\overline{\Omega_K}$ and $G_{\Omega_K}(\zeta) = 0$ for $\zeta \in \partial\Omega_K$;*

*(3) $G_{\Omega_K}(z^{-1}) - \log|z|$ extends to a harmonic function near $z = 0$.*

*Proof.* The uniqueness follows from the maximum principle (try!). To prove the existence, take $z_0 \in \mathbb{C} \setminus \overline{\Omega_K}$ and consider

$$D := \{0\} \cup \{z \in \mathbb{C} \setminus \{0\} : z^{-1} + z_0 \in \Omega_K\}.$$

We know that $D$ is a bounded domain in $\mathbb{C}$ with analytic boundary and $0 \in D$. One may check that

$$G_{\Omega_K}(\xi) := G_D\left(\frac{1}{\xi - z_0}, 0\right)$$

satisfies (1), (2) and (3). (11th Mar) □

**Definition 26.** *Let $K$ be a compact set in $\mathbb{C}$. Assume that $\Omega_K$ has real analytic boundary. Then we call $G_{\Omega_K}$ in Proposition 11 the* ***Green function of $\Omega_K$ with a pole at $\infty$***. *The following limit*

$$\gamma := \lim_{z \to 0}\left(G_{\Omega_K}(z^{-1}) - \log|z|\right) \tag{5.9}$$

*is called the* ***Robin constant*** *of $K$.*

We shall use the following result in the next subsection.

**Lemma 2.** *Let $K$ be a compact set in $\mathbb{C}$. Assume that $\Omega_K$ has real analytic boundary. Put*

$$p(\zeta,\infty) := \frac{1}{2\pi}\frac{\partial G_{\Omega_K}(\zeta)}{\partial n_\zeta}, \tag{5.10}$$

*where $n_\zeta$ denotes the outward unit normal vector at $\zeta \in \partial\Omega_K$. Then*

$$d\mu_K(\zeta) := p(\zeta,\infty)|d\zeta|, \ \ \zeta \in \partial\Omega_K, \tag{5.11}$$

*defines a probability measure $\mu_K$ on $K$ supported on $\partial\Omega_K$ satisfying*

$$p_{\mu_K}(z) := \int_K \log|z-\zeta|\, d\mu_K(\zeta) = \begin{cases} \gamma - G_{\Omega_K}(z) & z \in \Omega_K \\ \gamma & z \in \mathbb{C}\setminus\Omega_K. \end{cases} \tag{5.12}$$

*Proof.* Similar to (5.4), we have

$$u(\infty) = \frac{1}{2\pi}\int_{\Omega_K} G_{\Omega_K}(\zeta)\Delta u(\zeta)\, dxdy + \frac{1}{2\pi}\int_{|\partial\Omega_K|} \frac{\partial G_{\Omega_K}(\zeta)}{\partial n_\zeta} u(\zeta)\, |d\zeta|, \ \ w \in \Omega, \tag{5.13}$$

where $|\partial\Omega_K|$ denotes the set $\partial\Omega_K$ with positive orientation (anticlockwise), for all function $u$ smooth on a neighborhood of $\bar\Omega_K$ such that $u(1/z)$ extends to a smooth function near $z=0$, here $u(\infty) := \lim_{z\to 0} u(1/z)$. Thus we know that $\mu_K$ is a probability measure supported on $\partial\Omega_K$ in $K$. Hence it suffices to prove (5.12). Since $\log|z-\zeta|$ is harmonic outside $\zeta$, we know that

$$p_{\mu_K}(z) = \frac{1}{2\pi}\int_{\partial\Omega_K} \log|z-\zeta|\, \frac{\partial G_{\Omega_K}(\zeta)}{\partial n_\zeta}\, |d\zeta|$$

is harmonic outside $\partial\Omega_K$. For fixed $z \in \mathbb{C}\setminus\overline{\Omega_K}$, apply (5.13) to

$$u(\zeta) := \log|z-\zeta| + G_{\Omega_K}(\zeta),$$

we get

$$p_{\mu_K}(z) = u(\infty) = \gamma.$$

Since $\partial\Omega_K$ is analytic and $G_{\Omega_K}$ is harmonic near $\partial\Omega_K$, we know that $p_{\mu_K}$ is continuous near $\partial\Omega_K$ (see Exercise 4 below), thus $p_{\mu_K}(z) = \gamma$ also for $z \in \partial\Omega_K$. Now it remains to show that $p_{\mu_K} = \gamma - G_{\Omega_K}$ on $\Omega_K$, we already know that they are harmonic on $\Omega_K$, equal on $\partial\Omega_K$ and

$$\lim_{z\to\infty}(p_{\mu_K}(z) - \log|z|) = 0 = \lim_{z\to\infty}(\gamma - G_{\Omega_K}(z) - \log|z|),$$

so they must equal on $\Omega_K$ by the maximum principle. □

**Definition 27.** *Let $K$ be a compact set in $\mathbb{C}$. Assume that $\Omega_K$ has real analytic boundary. Then we call $\mu_K$ defined in (5.11) the **equilibrium measure** on $K$. We also call the function $p_{\mu_K}$ in (5.12) the **equilibrium potential** of $K$.*

**Definition 28.** *The **(logarithmic) capacity** of a compact set $K \subset \mathbb{C}$ is defined by*

$$c(K) := \inf_{n\geq 1,\, a_1,\cdots,a_n\in\mathbb{C}} \sup_{z\in K} |z^n + a_1 z^{n-1} + \cdots + a_n|^{1/n}.$$

In section 7, we shall follow [A1, section 2.2] to study the extremal properties of $\mu_K$ and prove that $e^\gamma = c(K)$ (recall that $\gamma$ denotes the Robin constant of $K$).

*Exercise 1*: Show that if $S$ is a finite set (means it has finite elements, can be empty) in $\mathbb{C}$. Then $\mathbb{C} \setminus S$ is parabolic.

*Solution*: Otherwise $\mathbb{C} \setminus S$ is hyperbolic and we know that there exists a subharmonic function $\psi \leq 0$ on $\mathbb{C} \setminus S$ with a simple pole at some $w \in \mathbb{C} \setminus S$ (here we use Definition 32 for subharmonic functions, in particular $\log |z - w|$ is subharmonic also for $z$ near $w$). Since $\psi \leq 0$, we know that $\psi$ extends (try!, see the proof of Theorem 3.6.1 in [Ra, page 67]) to a subharmonic function (still denoted by $\psi$) on $\mathbb{C} \cup \infty$. Then the maximal principle implies that $\psi$ is a constant, which contradicts to the fact that $\psi$ has a simple pole.

*Exercise 2*: Compute the Robin constant of the circle $|z - a| = r$ and prove the following formula
$$\frac{1}{2\pi} \int_0^{2\pi} \log |z - a - re^{i\theta}| \, d\theta = \max\{\log r, \log |z - a|\}$$
for the equilibrium potential of the circle $|z - a| = r$.

*Solution*: Denote by $K$ the circle $|z - a| = r$, we know that
$$G_{\Omega_K}(\zeta) = \log |\frac{r}{\zeta - a}|.$$
Hence the Robin constant $\gamma$ of the circle $|z - a| = r$ is
$$\gamma = \lim_{z \to 0} \{\log |\frac{r}{1/z - a}| - \log |z|\} = \log r.$$
By Exercise 2 in page 30, the equilibrium measure of $|z - a| = r$ is
$$d\mu_K = \frac{1}{2\pi} \frac{\partial G_{\Omega_K}(\zeta)}{\partial n_\zeta} |d\zeta| = \frac{1}{2\pi} d\theta.$$
Thus
$$p_{\mu_K}(z) = \int_{|w-a|=r} \log |z - w| \, \frac{1}{2\pi} d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log |z - a - re^{i\theta}| \, d\theta.$$
Hence (5.12) gives
$$p_{\mu_K}(z) = \max\{\log r, \log |z - a|\}.$$

*Exercise 3*: Prove (5.13). *Hint: Note that $|\partial \Omega_K| = -\partial \Omega_K$. Apply Green's formula to $D_R := \Omega_K \cap \{|z| < R\}$ and let $R \to \infty$.*

*Solution*: By Green's formula, we have
$$\int_{D_R} G_{\Omega_K}(\zeta) \Delta u(\zeta) - u(\zeta) \Delta G_{\Omega_K}(\zeta) \, dxdy = \int_{\partial D_R} G_{\Omega_K}(\zeta) \frac{\partial u(\zeta)}{\partial n_\zeta} - \frac{\partial G_{\Omega_K}(\zeta)}{\partial n_\zeta} u(\zeta) \, |d\zeta|$$
where $D_R := \Omega_K \cap \{|z| < R\}$. Note that $\partial D_R = \{|\zeta| = R\} \cup \partial \Omega_K$, hence
$$\int_{\partial D_R} G_{\Omega_K}(\zeta) \frac{\partial u(\zeta)}{\partial n_\zeta} |d\zeta| = \int_{|\zeta|=R} G_{\Omega_K}(\zeta) \frac{\partial u(\zeta)}{\partial n_\zeta} |d\zeta| = \int_0^{2\pi} G_{\Omega_K}(Re^{i\theta}) R u_r(Re^{i\theta}) d\theta.$$

Recall that $g(z) = u(1/z)$ is smooth near $z = 0$, hence
$$g(r^{-1}e^{-i\theta}) = u(re^{i\theta})$$
gives
$$u_r = g_x \frac{\cos\theta}{-r^2} + g_y \frac{\sin\theta}{r^2}.$$
Thus $|u_r(Re^{i\theta})| \leq CR^{-2}$ and we have
$$\lim_{R\to\infty} \int_{\partial D_R} G_{\Omega_K}(\zeta) \frac{\partial u(\zeta)}{\partial n_\zeta} |d\zeta| = \lim_{R\to\infty} R^{-1}\log R = 0.$$
Hence Green's formula reduces to
$$\int_{\Omega_K} G_{\Omega_K}(\zeta)\Delta u(\zeta)\, dxdy = -\lim_{R\to\infty} \int_{|z|=R} \frac{\partial G_{\Omega_K}(\zeta)}{\partial n_\zeta}\, u(\zeta)\, |d\zeta| - \int_{|\partial\Omega_K|} \frac{\partial G_{\Omega_K}(\zeta)}{\partial n_\zeta}\, u(\zeta)\, |d\zeta|,$$
where $n_\zeta$ denotes the outer normal vector for $D_R$. It suffices to check
$$-\lim_{R\to\infty} \int_{|z|=R} \frac{\partial G_{\Omega_K}(\zeta)}{\partial n_\zeta}\, u(\zeta)\, |d\zeta| = 2\pi u(\infty). \tag{5.14}$$
Put
$$\psi(\zeta) = G_{\Omega_K}(\zeta) + \log|\zeta|,$$
we know that $\psi$ is smooth at $\infty$, thus
$$-\lim_{R\to\infty} \int_{|z|=R} \frac{\partial G_{\Omega_K}(\zeta)}{\partial n_\zeta}\, u(\zeta)\, |d\zeta| = \lim_{R\to\infty} \int_{|z|=R} \frac{\partial \log|\zeta|}{\partial n_\zeta}\, u(\zeta)\, |d\zeta| = \lim_{R\to\infty} \int_{|z|=R} u(\zeta)\, d\theta$$
gives (5.14).

*Exercise 4*: Show that $P_{\mu_K}$ defined in (5.12) is continuous near $\partial\Omega_K$. *Hint: recall that we assume that $\Omega_K$ has analytic boundary. Then it is enough to show that*
$$\lim_{z\to 0,\, z\in\mathbb{C}} \int_{-1}^{1} (\log|\zeta - z|)\, f(\zeta)\, d\zeta = \int_{-1}^{1} (\log|\zeta|)\, f(\zeta)\, d\zeta \tag{5.15}$$
*is continuous, where $f$ is a smooth function on $\mathbb{R}$.*

*Solution*: Note that for every small $\varepsilon > 0$, we have
$$\lim_{z\to 0,\, z\in\mathbb{C}} \int_{\varepsilon<|\zeta|<1} (\log|\zeta - z|)\, f(\zeta)\, d\zeta = \int_{\varepsilon<|\zeta|<1} (\log|\zeta|)\, f(\zeta)\, d\zeta. \tag{5.16}$$
On the other hand, for $C := \sup_{|\zeta|<1}|f|$, we have
$$|\int_{|\zeta|\leq\varepsilon} (\log|\zeta - z|)\, f(\zeta)\, d\zeta| \leq C\int_{|\zeta|\leq\varepsilon} -\log(\zeta - \mathrm{Re}\, z)\, d\zeta \leq C\int_{|\zeta|\leq\varepsilon+|z|} -\log\zeta\, d\zeta,$$
which gives
$$\limsup_{z\to 0} |\int_{|\zeta|\leq\varepsilon} (\log|\zeta - z|)\, f(\zeta)\, d\zeta| \leq C\int_{|\zeta|\leq\varepsilon} -\log\zeta\, d\zeta = 2C\varepsilon(1 - \log\varepsilon). \tag{5.17}$$
Letting $\varepsilon \to 0$, we know that (5.15) follows from (5.16) and (5.17).

## 6. A SHORT COURSE ON BOREL MEASURES

6.1. **Riesz representation theorem.** We shall mainly follow the Ransford book [Ra] in this part. Let $X$ be a topological space. A function $\phi : X \to \mathbb{R}$ is said to be continuous if

$$\phi^{-1}(a,b) := \{x \in X : a < \phi(x) < b\}$$

is open for all $a, b$ in $\mathbb{R}$. The support of $\phi$ is defined as

$$\operatorname{supp}\phi := \overline{\{x \in X : \phi(x) \neq 0\}}.$$

The space of all continuous functions with compact support in $X$ is denoted by $C_c(X)$. The fundamental theorem in the Borel measure theory is the following metric space version of the Riesz representation theorem. Recall that $(X, d)$ is called a metric space if $d$ is a non-negative function on $X \times X$ such that

$$d(x,y) = d(y,x), \ d(x,y) + d(y,z) \geq d(x,z),$$

for all $x, y, z$ in $X$ and $d(x,y) > 0$ for all $x \neq y$. A set $U$ in $(X, d)$ is said to be open if for every $x \in U$ we have

$$B_r(x) := \{y \in X : d(x,y) < r\} \subset U$$

for some $r > 0$. These open sets give a natural topology on $X$. Thus the notion of compact set is well defined on $(X, d)$. We say that $X$ has compact exhaustion if

$$X = \cup_{n \geq 1} K_n,$$

where each $K_n$ is compact in $X$ and $K_n \subset K_{n+1}^\circ$, where $K_{n+1}^\circ$ denotes the interior of $K_{n+1}$ (i.e. the largest open set in $K_{n+1}$). A typical example is $\mathbb{C}$ with the euclidean metric. (12th Mar)

**Theorem 44.** *Let $(X, d)$ be a metric space with compact exhaustion. Let*

$$\Lambda : C_c(X) \to \mathbb{R}$$

*be an $\mathbb{R}$-linear mapping. If $\Lambda$ is positive (i.e. $\Lambda(\phi) \geq 0$ for all $\phi \geq 0$) then there exists a unique Borel measure $\mu$ on $X$ such that $\mu(K) < \infty$ for all compact $K \subset X$ and*

$$\Lambda(\phi) = \int_X \phi \, d\mu, \quad \forall\, \phi \in C_c(X).$$

*(notions related to the Borel measure will be given in the proof).*

*Proof. Notion:* We write compact $K \prec \phi$ if

$$\phi \in C_c(X), \ \ 0 \leq \phi \leq 1, \ \ \phi = 1 \text{ on } K;$$

we write $\phi \prec U$ open if

$$\phi \in C_c(X), \ \ 0 \leq \phi \leq 1, \ \ \operatorname{supp}\phi \subset U.$$

*Definition:* For $U$ open, we define

$$\mu^*(U) := \sup\{\Lambda(\phi) : \phi \prec U\}.$$

For an arbitrary subset $E$ of $X$, we define

$$\mu^*(E) := \inf\{\mu^*(U) : \text{open } U \supset E\},$$

*Step 1:* Check that $\mu^*: 2^X \to [0,\infty]$ ($2^X$ denotes the collection of all subsets of $X$) is an outer measure (see page 213 in [Ra] for details), i.e.

(1) $\mu^*(\emptyset)=0$; ($\phi \prec \emptyset$ means $\phi=0$)

(2) $\mu^*(E_1)\leq \mu^*(E_2)$ for all $E_1\subset E_2\subset X$; (obvious)

(3) $\mu^*(\cup_n E_n)\leq \sum_n \mu^*(E_n)$ for all $E_n\subset X, n=1,2,\cdots$. Proof: take $U_n$ such that $\mu^*(E_n)\geq \mu^*(U_n)-2^{-n}\varepsilon$, observe that $\mu^*(\cup_n E_n)\leq \mu^*(\cup_n U_n)$. If $\phi\prec \cup_n U_n$ then compactness of $\operatorname{supp}\phi$ gives $\phi\prec U_1\cup\cdots U_N$ for some $N$ and there exists open $V_n$ with compact $\overline{V_n}\subset U_n$ such that $\phi\prec V_1\cup\cdots V_N$ (here we use the fact that $X$ has a compact exhaustion). Then

$$\phi=\sum_{n=1}^N \phi_n, \quad \phi_n(x)=\frac{\phi(x)\, d(x,V_n^c)}{d(x,\operatorname{supp}\phi)+\sum_{n=1}^N d(x,V_n^c)}$$

gives $\Lambda(\phi)\leq \sum_{n=1}^N \Lambda(\phi_n)\leq \sum_n \mu^*(U_n)\leq \sum_n \mu^*(E_n)+\varepsilon$. Thus (3) follows.

*Definition:* We say that $A\subset X$ is $\mu^*$-measurable if

$$\mu^*(E)=\mu^*(E\cap A)+\mu^*(E\cap A^c) \ \text{ for all } E\subset X,$$

where $A^c$ denotes the complement of $A$.

*Carathéodory's Lemma:*

(1) The collection, say $\mathcal{M}$, of all $\mu^*$-measurable sets forms a $\sigma$-algebra (i.e. $\emptyset\in\mathcal{M}$; $A\in\mathcal{M}$ implies $A^c\in\mathcal{M}$ and $A_j\in\mathcal{M}$, $j\geq 1$, implies both $\cap_{j=1}^\infty A_j\in\mathcal{M}$ and $\cup_{j=1}^\infty A_j\in\mathcal{M}$).

(2) The restriction

$$\mu^*:\mathcal{M}\to[0,\infty]$$

of $\mu^*$ to $\mathcal{M}$ defines a measure (we call it the generalized Lebesgue measure) on $(X,\mathcal{M})$ (i.e. $\mu^*(\emptyset)=0$ and $\mu^*(\cup_{j=1}^\infty A_j)=\sum_{j=1}^\infty \mu^*(A_j)$ for all $A_j\in\mathcal{M}$ with $A_j\cap A_k=\emptyset$ for $j\neq k$).

*Proof of Carathéodory's Lemma.* The first two properties in (1) are trivial. For $A,B\in\mathcal{M}$ and $E\subset X$, we have $\mu^*(E)=\mu^*(E\cap A)+\mu^*(E\cap A^c)$ since $A\in\mathcal{M}$. Now $B\in\mathcal{M}$ further gives

$$\mu^*(E)=\mu^*(E\cap A\cap B)+\mu^*(E\cap A\cap B^c)+\mu^*(E\cap A^c\cap B)+\mu^*(E\cap A^c\cap B^c).$$

Now $A\in\mathcal{M}$ also gives $\mu^*(E\cap(A\cap B)^c)=\mu^*(E\cap(A\cap B)^c\cap A)+\mu^*(E\cap(A\cap B)^c\cap A^c)$, hence we have

$$\mu^*(E\cap(A\cap B)^c)=\mu^*(E\cap A\cap B^c)+\mu^*(E\cap A^c).$$

Since $B\in\mathcal{M}$, we further have

$$\mu^*(E\cap A^c)=\mu^*(E\cap A^c\cap B)+\mu^*(E\cap A^c\cap B^c).$$

Thus we obtain $\mu^*(E)=\mu^*(E\cap A\cap B)+\mu^*(E\cap(A\cap B)^c)$ and $A\cap B\in\mathcal{M}$. Hence any finite union and any finite intersection of measurable sets is measurable. For the remaining part of the proof, see [LL, Page 29-31]. (18th Mar) □

*Step 2:* Check that every open set $U \in \mathcal{M}$ (see [Ra, Page 214]). Hence the Borel $\sigma$-algebra (i.e. the minimal $\sigma$-algebra that contains all open sets) $\mathcal{B} \subset \mathcal{M}$. We shall denote the restriction of $\mu^*$ to $\mathcal{B}$ by $\mu$

$$\mu : \mathcal{B} \to [0, \infty]$$
$$A \mapsto \mu^*(A).$$

Thus $\mu$ is a Borel measure (i.e. a measure on the Borel $\sigma$-algebra).

*Step 3:* Check that (see [Ra, Page 214 and Lemma A.3.4 in Page 213] for the proof) for every compact $K \subset X$ (thus $K \in \mathcal{B}$)

$$\mu(K) = \inf\{\Lambda(\phi) : K \prec \phi\},$$

in particular $\mu(K) < \infty$.

**Definition 29.** *A function $f : X \to [-\infty, \infty]$ is said to be Borel measurable (or simply, Borel) if*

$$f^{-1}(-\infty, a) := \{x \in X : f(x) < a\} \in \mathcal{B}, \ \ \forall\, a \in \mathbb{R}.$$

One may check that (see Exercise 18 in page 39 of [LL]) if $f, g$ are Borel then $af + bg$, $fg$, $f/g$ are Borel for all $a, b \in \mathbb{R}$. Moreover, if $f_k$ are Borel then $\sup_{k\geq 1} f_k$, $\liminf_{k\to\infty} f_k$ are Borel.

*Example:* Simple functions

$$s := \sum_{j=1}^{N} a_j\, 1_{A_j}, \quad A_j \in \mathcal{B}, \quad a_j \geq 0,\ 1 \leq j \leq N,$$

are Borel, where $1_{A_j} = 1$ on $A_j$ and equals zero elsewhere. We define the integral of $s$ as

$$\int_X s\, d\mu := \sum_{j=1}^{N} a_j\, \mu(A_j) \ (a_j \geq 0 \text{ to make sure that the sum is well defined}).$$

For a general function $f$ with $f \geq 0$ on $X$, for some $c \in \mathbb{R}$, we define its integral as

$$\int_X f\, d\mu = \sup\left\{ \int_X s\, d\mu : \text{simple } s \leq f \right\}. \tag{6.1}$$

For a general function $f$ on $X$, we say that the integral of $f$ is well defined if $\int_X |f|\, d\mu < \infty$, in which case, we define

$$\int_X f\, d\mu := \int_X \max\{f, 0\}\, d\mu - \int_X \max\{-f, 0\}\, d\mu.$$

We observe that $\int_X \phi\, d\mu$ is always well defined for $\phi \in C_c(X)$.

*Step 4:* Check that $\Lambda(\phi) = \int_X \phi\, d\mu$ and $\mu$ is unique (see [Ra, Page 212, 215]). □

**Remark**: *By* [Ra, Lemma A.3.3], *we also have*

$$\mu(A) = \sup\{\mu(K) : \text{ compact } K \subset A\}, \ \ \forall\, A \in \mathcal{B}.$$

*A standard example of* $\Lambda$ *is:* $X = \mathbb{C}$ *with*

$$\Lambda(\phi) := \int_{\mathbb{C}} \phi(x+iy)\, dxdy,$$

*in this case,* $\mathcal{M}$ *defined in the Carathéodory lemma is usually called the Lebesgue* $\sigma$*-algebra (so* $A \in \mathcal{M}$ *is said to be Lebesgue measurable). Hence* $\mu$ *is the restriction to* $\mathcal{B}$ *of the Lebesgue measure. A second example is* $X = \mathbb{C}$ *with*

$$\Lambda(\phi) := \int_0^{2\pi} \phi(e^{i\theta})\, d\theta.$$

*In this case, the corresponding Borel measure* $\mu$ *is supported on the unit circle* $|z| = 1$*, where the support of* $\mu$ *is defined by*

$$\operatorname{supp}\mu := \cap\{F : F \subset X \text{ is closed with } \mu(X \setminus F) = 0\}. \tag{6.2}$$

6.2. **Integral of a Borel function with respect to a Borel measure.** The integral in (6.1) is not so easy to compute/use, but it is known (see [LL, Page 32, 33]) that if $f \geq 0$ is Borel then (see [LL, Page 34] for the proof of a stronger version in case the right hand side is finite)

$$\int_X f\, d\mu = \int_0^\infty \mu\{f(x) > t\}\, dt. \tag{6.3}$$

The right hand side is the Riemann integral of a *decreasing* (thus continuous outside a countable set) function on $[0, \infty)$.

**Theorem 45** (Fatou's lemma)**.** *If* $f_j \geq 0$ *are Borel then (see* [LL, Page 18] *for the proof)*

$$\liminf_{j\to\infty} \int_X f_j\, d\mu \geq \int_X \liminf_{j\to\infty} f_j\, d\mu.$$

**Definition 30.** *A complex function* $f : X \to \mathbb{C}$ *is said to be* ***Borel*** *if both its real and imaginary parts are Borel. A complex Borel function* $f$ *is said to be* ***integrable*** *if*

$$\int_X |f|\, d\mu < \infty.$$

*The* $L^p$ ***space****,* $1 \leq p < \infty$*, is defined by (see* [Ax, Chapter 7]*)*

$$L^p(X,\mu) := \left\{ \text{complex Borel } f : \int_X |f|^p\, d\mu < \infty \right\} / \sim,$$

*where* $\sim$ *means we identify functions which are equal outside a* $\mu$*-measure zero set.*

**Remark**: *It is known that each* $L^p(X,\mu)$ *is a complex Banach space (i.e. complete complex normed space, see* [Ru, page 67] *and* [LL, page 52]*) and* $C_c(X)$ *is dense in* $L^p(X,\mu)$ *(see* [Ru, Page 69]*). People often identify* $f$ *with its equivalent class in* $L^p(X,\mu)$*. In this way,* $f = g$ *means they are equal outside a* $\mu$*-measure zero set, and* $f_j \to f$ *pointwise on* $X$ *means* $f_j(x) \to f(x)$ *for all* $x$ *outside a* $\mu$*-measure zero set. (19th Mar)*

**Theorem 46** (Monotone convergence). *If $f_j \in L^1(X,\mu)$ are real and $f_1 \le f_2 \le \cdots$, then*

$$\lim_{j\to\infty}\int_X f_j\, d\mu = \int_X \lim_{j\to\infty} f_j\, d\mu,$$

*see* [LL, page 17] *for the proof.*

**Theorem 47** (Dominated convergence). *If $f_j \in L^1(X,\mu)$ pointwise converge to $f$ and there exists $G_j,\, G \in L^1(X,\mu)$ with $|f_j| \le G_j$ for all $j$ and*

$$\lim_{j\to\infty}\int_X |G_j - G|\, d\mu = 0,$$

*then $f \in L^1(X,\mu)$ and*

$$\lim_{j\to\infty}\int_X f_j\, d\mu = \int_X \lim_{j\to\infty} f_j\, d\mu,$$

*see* [LL, page 19, 20] *for the proof.*

6.3. **Complex Borel measures.**

**Definition 31.** *A* ***complex Borel measure*** *is a function*

$$\mu : \mathcal{B} \to \mathbb{C}$$

*such that $\mu(\emptyset) = 0$ and*

$$\sum_{j=1}^{\infty} |\mu(A_j)| < \infty, \quad \sum_{j=1}^{\infty} \mu(A_j) = \mu(\cup_{j=1}^{\infty} A_j)$$

*for all disjoint set $A_j \in \mathcal{B}$.*

**Remark**: *If $\mu$ is a complex measure, then its variation $|\mu|$ defined by*

$$|\mu|(A) := \sup \sum_j |\mu(A_j)|,$$

*where the supremum runs over all sequences of disjoint Borel sets $A_j$ whose union is $A$, is a Borel measure. We also know that its total variation $||\mu|| := |\mu|(X) < \infty$. We have the following Riesz Representation theorem for complex Borel measures (see Theorem 6.19 in page 130 in* [Ru] *for the statement and the proof, we do not need "regular" because all finite Borel measures on a metric space is regular, see Theorem A.2.2 in* [Ra]*.)*

**Theorem 48.** *Let $X$ be a metric space with compact exhaustion. Let*

$$\Lambda : C_c(X,\mathbb{C}) \to \mathbb{C}$$

*be a $\mathbb{C}$-linear mapping. If*

$$||\Lambda|| := \sup\left\{ |\Lambda(\phi)| : \phi \in C_c(X,\mathbb{C}),\ \sup_X |\phi| = 1 \right\} < \infty,$$

*then there exists a unique complex Borel measure $\mu$ on $X$ such that*

$$\Lambda(\phi) = \int_X \phi \, d\mu, \quad \forall \, \phi \in C_c(X).$$

*Moreover, we have $||\mu|| = ||\Lambda||$.*

6.4. **Fubini theorem.** Let $X_1, X_2$ be metric spaces with compact exhaustion. Let $\mu_j$ be Borel measures on $X_j$ such that $\mu_j(K_j) < \infty$ for every compact $K_j \subset X_j$, $j = 1, 2$. The Borel $\sigma$-algebra $\mathcal{B}_{X_1 \times X_2}$ on $X_1 \times X_2$ equals the smallest $\sigma$-algebra, say $\mathcal{B}_{X_1} \times \mathcal{B}_{X_2}$, containing $A_1 \times A_2$ for all $A_1 \in \mathcal{B}_{X_1}$ and $A_2 \in \mathcal{B}_{X_2}$. It is known that (see page 11 in [LL] for the uniqueness and page 23 in [LL] for the existence) there exists a unique measure $\mu$ on $\mathcal{B}_{X_1 \times X_2}$ such that

$$\mu(A_1 \times A_2) = \mu_1(A_1) \times \mu_2(A_2), \quad \forall \, A_1 \in \mathcal{B}_{X_1}, \ A_2 \in \mathcal{B}_{X_2}.$$

We write $\mu := \mu_1 \times \mu_2$. We have the following Fubini's theorem (see [LL, page 25]).

**Theorem 49.** *With the notation above, if $f \geq 0$ is Borel on $X_1 \times X_2$ then*

$$\int_{X_1 \times X_2} f \, d(\mu_1 \times \mu_2) = \int_{X_1} \left( \int_{X_2} f \, d\mu_2 \right) d\mu_1 = \int_{X_2} \left( \int_{X_1} f \, d\mu_1 \right) d\mu_2. \tag{6.4}$$

*If $f$ is complex valued then* (6.4) *holds if one assumes in addition that*

$$\int_{X_1 \times X_2} |f| \, d(\mu_1 \times \mu_2) < \infty,$$

*(see* [LL, page 25] *for the proof and generalizations).*

*Reading task 1*: Read [LL, page 29-31] for Carathéodory's Lemma used in Step 1 of the proof of the Riesz representation theorem and read page 213-214 in [Ra] to complete the proof in Step 2 and Step 3.

*Reading task 2*: Read [LL, page 12-19, 32-34] for (6.3), Fatou's lemma, Monotone convergence theorem and Dominated convergence theorem.

*Reading task 3*: Read [LL, page 25] for the Fubini theorem.

*Reading task 4*: Read [Ru, page 130] for Theorem 48.

*Exercise 1*: Let $X$ be a topological space. A function $\phi : X \to [-\infty, \infty)$ is said to be *upper semicontinuous (usc)* if $\{x \in X : \phi(z) < c\}$ is open for every $c \in \mathbb{R}$.

(a) Assume that $\phi$ is usc on $X$. Let $K$ be a compact set. Show that there exists $z_0 \in K$ such that

$$\phi(z_0) = \sup_{z \in K} \phi(z).$$

*Solution*: Otherwise, put $C := \sup_{z \in K} \phi(z)$, we would have $\phi(z) < C$ for every $z \in K$, thus

$$K \subset \cup_{n \geq 1} U_n, \quad U_n := \{z \in X : \phi(z) < C - 1/n\}.$$

Since $\phi$ is usc, we know that each $U_n$ is open. Thus the compactness of $K$ implies that there exists $N$ such that

$$K \subset \cup_{N \geq n \geq 1} U_n = U_N,$$

which implies $C \leq C - 1/N$, a contradiction.

(b) Show that usc functions are Borel measurable.

*Solution*: Follows from the definition, since every open set is Borel.

(c) Let $v$ be usc on a domain $\Omega \subset \mathbb{C}$. Let $|z - z_0| = r$ be a circle inside $\Omega$. Put

$$v^+ := \max\{v, 0\}, \quad v^- := \max\{-v, 0\}.$$

Use (b) to prove that

$$\int_0^{2\pi} v^+(z_0 + re^{i\theta})\, d\theta < \infty.$$

—in this way, the integral of $v$ over the circle can be defined as

$$\int_0^{2\pi} v(z_0 + re^{i\theta})\, d\theta := \int_0^{2\pi} v^+(z_0 + re^{i\theta})\, d\theta - \int_0^{2\pi} v^-(z_0 + re^{i\theta})\, d\theta \in [-\infty, \infty).$$

*Solution*: One may check that $v^+$ is usc, thus by (b), we know that $v^+$ is Borel measurable, hence the integral of $v^+$ over the circle $|z - z_0| = r$ is well defined. Since the circle is compact, we know that $v^+ \leq C$ for some $C$ on $|z - z_0| = r$, which gives

$$\int_0^{2\pi} v^+(z_0 + re^{i\theta})\, d\theta \leq 2\pi C < \infty.$$

*Exercise 2*: With the definition of usc on Exercise 1.

(a) Assume that $v_1, v_2$ are usc, show that $c_1 v_1 + c_2 v_2$ and $\max\{v_1, v_2\}$ are usc, where $c_1, c_2$ are positive constants.

*Solution*: Since $c_1, c_2 > 0$, we have (try!)

$$\{c_1 v_1 + c_2 v_2 < a\} = \cup_{b \in \mathbb{R}} \{v_1 < b\} \cap \{v_2 < (a - c_1 b)/c_2\}$$

is open for every $a \in \mathbb{R}$, thus $c_1 v_1 + c_2 v_2$ is usc. Similarly,

$$\{\max\{v_1, v_2\} < a\} = \{v_1 < a\} \cap \{v_2 < a\}$$

is also open and we know that $\max\{v_1, v_2\}$ is usc.

(b) Let $\Omega$ be a domain in $\mathbb{C}$. Show that $v$ is usc on $\Omega$ if and only if

$$\limsup_{z \to z_0} v(z) \leq v(z_0)$$

for every $z_0 \in \Omega$.

*Solution*: If $v$ is usc, then for every $\varepsilon > 0$

$$\{v < v(z_0) + \varepsilon\}$$

is a neighborhood of $z_0$, thus $\limsup_{z \to z_0} v(z) \leq v(z_0) + \varepsilon$ for for every $\varepsilon > 0$. Letting $\varepsilon \to 0$, we obtain $\limsup_{z \to z_0} v(z) \leq v(z_0)$. On the other hand, assume the condition in (b), we need to check that

$$U_c := \{v < c\}$$

is open for every $c \in \mathbb{R}$. Assume that $z_0 \in U_c$, then we know that $v(z_0) < c$, take $\varepsilon := (c - v(z_0))/2$, we know that $v(z_0) < c - \varepsilon$, and the condition in (b) implies that $v(z) < c$ for $|z - z_0| < r$, where $r$ is sufficiently small. Thus $U_c$ is open.

(c) Show that $\log |h|$ is usc for every holomorphic function $h$ on $\Omega$ and (with respect to the definition in Exercise 1 (c))

$$\log |f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log |f(z_0 + re^{i\theta})| \, d\theta,$$

where $f$ is holomorphic on a neighborhood of $|z - z_0| \leq r$.

*Solution*: Continuity of $|h|$ implies that $\{\log |h| < c\} = \{|h| < e^c\}$ is open for every $c \in \mathbb{R}$, thus $\log |h|$ is usc. To verify the submean inequality, one may check that

$$v_\varepsilon := \frac{1}{2} \log(|f|^2 + \varepsilon)$$

is a smooth subharmonic function for every $\varepsilon \geq 0$ (try to show that $(v_\varepsilon)_{z\bar{z}} \geq 0$). Each $v_\varepsilon$ satisfies the submean inequality, thus letting $\varepsilon \to 0$, we know that (try!) $\log |f|$ also satisfies the submean inequality.

*Exercise 3*: Let $\mu$ be a Borel probability measure on a compact set $K \subset \mathbb{C}$. Put

$$p_\mu(z) := \int_K \log |z - w| \, d\mu(w).$$

(a) Use Fatou's lemma to show that $p_\mu$ is usc on $\mathbb{C}$ and continuous on $\mathbb{C} \setminus \operatorname{supp} \mu$;

*Solution*: *Step 1: $p_\mu$ is usc.* By Exercise 2 (b), we need to show that

$$(\star 1) \quad \limsup_{z \to z_0} p_\mu(z) \leq p_\mu(z_0).$$

By a dilation transform, one may assume that both $z_0$ and $K$ lie in $\{|z| < 1/2\}$, then we know that $-\log |z - w| > 0$ for $z$ close to $z_0$ and $w \in K$. Thus Fatou's lemma implies that

$$\liminf_{z \to z_0} \int_K - \log |z - w| \, d\mu(w) \geq \int_K \liminf_{z \to z_0} - \log |z - w| \, d\mu(w) = -p_\mu(z_0),$$

hence $(\star 1)$ follows.

*Step 2*: *$p_\mu$ is continuous on $\mathbb{C} \setminus \operatorname{supp} \mu$.* We need to show that

$$(\star 2) \quad \liminf_{z \to z_0} p_\mu(z) \geq p_\mu(z_0),$$

for every $z_0 \in \mathbb{C} \setminus \operatorname{supp} \mu$. By a dilation transform, one may assume that the distance from $z_0$ to $\operatorname{supp} \mu$ is $> 1$. Then $\log |z - w| > 0$ for $z$ close to $z_0$ and $w \in K$. Thus Fatou's lemma implies that

$$\liminf_{z \to z_0} \int_K \log |z - w| \, d\mu(w) \geq \int_K \liminf_{z \to z_0} \log |z - w| \, d\mu(w) = p_\mu(z_0),$$

hence $(\star 2)$ follows.

(b) Use the Fubini theorem to show that $p_\mu$ satisfies the submean inequality on $\mathbb{C}$ and mean value property on $\mathbb{C} \setminus \operatorname{supp} \mu$.

*Solution*: By Exercise 2 (c), we know that for every $w \in \mathbb{C}$, $z \mapsto \log|z-w|$ satisfies the submean inequality, i.e.

$$\log|z_0 - w| \le \frac{1}{2\pi}\int_0^{2\pi} \log|z_0 + re^{i\theta} - w|\, d\theta,$$

hence we have

$$\frac{1}{2\pi}\int_0^{2\pi} p_\mu(z_0 + re^{i\theta})\, d\theta = \int_K \left(\frac{1}{2\pi}\int_0^{2\pi} \log|z_0 + re^{i\theta} - w|\, d\theta\right) d\mu(w) \ge p_\mu(z_0).$$

Thus $p_\mu$ satisfies the submean inequality on $\mathbb{C}$. Similarly, for every $w \in \operatorname{supp}\mu$, $z \mapsto \log|z-w|$ satisfies the mean-value property on $\mathbb{C} \setminus \operatorname{supp}\mu$, which implies that $p_\mu$ satisfies the mean value property on $\mathbb{C} \setminus \operatorname{supp}\mu$.

## 7. Extremal property of the equilibrium measure

### 7.1. General subharmonic functions.

**Definition 32.** *Let $\Omega$ be a domain in $\mathbb{C}$. We say that a function*

$$v : \Omega \to [-\infty, \infty)$$

*is **upper semicontinuous (usc)** if*

$$\limsup_{z \to z_0} v(z) \le v(z_0)$$

*for every $z_0 \in \Omega$. An usc function $v$ on $\Omega$ is said to be **subharmonic** if it satisfies the **local submean inequality**, i.e. for every $z_0 \in \Omega$, there exists $r_0 > 0$ with $|z - z_0| < r_0$ contained in $\Omega$ such that the following*

$$v(z_0) \le \frac{1}{2\pi}\int_0^{2\pi} v(z_0 + re^{i\theta})\, d\theta \tag{7.1}$$

*holds for every $0 < r < r_0$.*

**Remark 1**. *Note thet $v$ is usc if and only if $\{v < c\}$ is open for every $c \in \mathbb{R}$, hence subharmonic functions are always Borel (see Definition 29). Thus the integral in (7.1) is well defined.*

**Remark 2**. *An interesting class of non-continuous subharmonic functions is $\log|f|$ (see Theorem* [Ra, Theorem 2.2]*), where $f$ is holomorphic.*

**Remark 3**. *One may check that Theorem 33, Proposition 7, Proposition 6 and Theorem 34 also apply to non-continuous subharmonic functions.*

### 7.2. Potential of a Borel measure.

In potential theory, we shall study a class of subharmonic functions called "potentials" (generalization of (5.12)).

**Definition 33.** *Let $\mu$ be a Borel probability measure on a compact set $K \subset \mathbb{C}$. Its **potential** is the function $p_\mu : \mathbb{C} \to [-\infty, \infty)$ defined by*

$$p_\mu(z) := \int_K \log|z - w|\, d\mu(w). \tag{7.2}$$

**Remark.** *The potential $p_\mu$ is a natural generalization of $\frac{1}{n}\log|P_n|$, where $P_n$ is a monic degree $n$ polynomial with zeros lie in $K$. In fact, if we write*

$$P_n(z) = (z - z_1)\cdots(z - z_n), \quad z_j \in K, \quad 1 \le j \le n,$$

*then*

$$\frac{1}{n}\log|P_n| = p_\mu, \text{ for } \mu := \frac{\delta_{z_1} + \cdots + \delta_{z_n}}{n},$$

*where $\mu$ is the Borel measure associated to the functional $\phi \mapsto \sum_{j=1}^n \phi(z_j)/n$ (see the Riesz representation theorem in Theorem 44.*

**Proposition 12.** *Let $\mu$ be a Borel probability measure on a compact set $K \subset \mathbb{C}$. Then $p_\mu$ is subharmonic on $\mathbb{C}$, harmonic on $\mathbb{C} \setminus \operatorname{supp}\mu$,*

$$\lim_{z\to 0} p_\mu(z^{-1}) + \log|z| = 0 \tag{7.3}$$

*and $p_\mu(z^{-1}) + \log|z|$ extends to a harmonic function near $z = 0$.*

*Proof.* Exercise 3 in page 60 implies that $p_\mu$ is subharmonic on $\mathbb{C}$, harmonic on $\mathbb{C} \setminus \operatorname{supp}\mu$. The remaining part follows from

$$p_\mu(z^{-1}) + \log|z| = \int_K \log|1 - zw|\, d\mu(w).$$

(Try! Note that $z \mapsto \log|1 - zw|$ is harmonic for $z$ near $0$ and $w \in K$). □

### 7.3. Energy of a Borel measure and extremal properties.

**Definition 34.** *Let $\mu, \nu$ be a Borel probability measures on a compact set $K \subset \mathbb{C}$. We shall define*

$$V_\mu := \inf_{z\in\mathbb{C}} p_\mu(z), \quad W_\mu := \sup_{z\in K} p_\mu(z), \quad I(\mu,\nu) := \int_K p_\mu\, d\nu. \tag{7.4}$$

*We call $I(\mu) := I(\mu,\mu)$ the **energy** of $\mu$.*

**Remark.** *Since $p_\mu$ is harmonic on $\mathbb{C} \setminus K$ and $p_\mu(z) \to \infty$ as $z \to \infty$, one may apply the maximum principle to conclude that $V_\mu = \inf_{z\in K} p_\mu(z)$.*

The main theorem in this section is the following extremal property of $\mu_K$ in (5.11).

**Theorem 50.** *Let $K$ be a compact set in $\mathbb{C}$. Assume that $\mathbb{C} \setminus K$ has real analytic boundary. Then*

$$\gamma = V_{\mu_K} = W_{\mu_K} = I(\mu_K) = \sup_\mu I(\mu) = \sup_\mu V_\mu = \inf_\mu W_\mu, \tag{7.5}$$

*where the supremum and infimum are taken over all Borel probability measures on $K$.*

*Proof.* $\gamma = V_{\mu_K} = W_{\mu_K} = I(\mu_K)$ follows directly from $p_{\mu_K} = \gamma$ on $K$ by (5.12). It remains to do the following two steps.

*Step 1:* $\gamma = \sup_\mu V_\mu = \inf_\mu W_\mu$. By Fubini's theorem and (5.12), we have

$$\int_{K\times K} \log|z - w|\, d\mu(z) d\mu_K(w) = \int_K p_\mu\, d\mu_K = \int_K p_{\mu_K}\, d\mu = \gamma, \tag{7.6}$$

for any Borel probability measure $\mu$ on $K$. Hence

$$V_\mu \le \int_K p_\mu \, d\mu_K = \gamma \le W_\mu$$

together with $\gamma = V_{\mu_K} = W_{\mu_K}$, we obtain $\gamma = \sup_\mu V_\mu = \inf_\mu W_\mu$.

*Step 2:* $\gamma = \sup_\mu I(\mu)$. It suffices to show $I(\mu) \le \gamma$ for any Borel probability measure $\mu$ on $K$. Note that (5.12) gives

$$I(\mu, \mu_K) = \int_K p_{\mu_K} \, d\mu = \gamma = I(\mu_K),$$

hence

$$I(\mu) = I(\mu - \mu_K) - I(\mu_K) + 2I(\mu, \mu_K) = \gamma + I(\mu - \mu_K),$$

and step 2 follows from the lemma below. Remark (optional). *A simple proof of $I(\mu - \mu_K) \le 0$ using the language of currents: write $u = p_\mu - p_{\mu_K}$note that $dd^c(p_\mu) = dd^c(\log|z| \star d\mu) = d\mu$ (where we choose $d^c$ such that $dd^c \log|z| = \delta_0$) implies that*

$$I(\mu - \mu_K) = \int_{\mathbb{C}} u \, dd^c u = \lim_{R\to\infty} \left( \int_{|z|=R} u \, d^c u - \int_{|z|<R} du \wedge d^c u \right).$$

*Since $u(z^{-1})$ is harmonic near $z = 0$ and vanishes at $z = 0$, we know*

$$\int_{|z|=R} u(z) \, d^c u(z) = \int_{|z|=R^{-1}} u(z^{-1}) \, d^c u(z^{-1}) \to 0$$

*as $R \to \infty$. Hence*

$$I(\mu - \mu_K) = -\int_{\mathbb{C}} du \wedge d^c u \le 0. \tag{7.7}$$

*This proof also implies that $I(\mu - \mu_K) = 0$ if and only if $\mu = \mu_K$.* □

**Lemma 3.** *$I(\mu - \mu_K) \le 0$. (15th Apr)*

*Proof.* We claim that (the proof is given later)

$$\frac{1}{2\pi} \int_{|z|<R} \frac{dxdy}{|z - z_1||z - z_2|} = \log R - \log|z_1 - z_2| + C + \varepsilon(z_1, z_2, R), \tag{7.8}$$

where $C$ is a constant and $\varepsilon(z_1, z_2, R) \to 0$ for $R \to \infty$, uniformly when $z_1, z_2$ are on a compact set. Integrating (7.8) with respect to $\mu - \mu_K$ yields

$$\lim_{R\to\infty} \int_{|z|<R} \left( \int_K \frac{d\mu(\zeta) - d\mu_K(\zeta)}{|z - \zeta|} \right)^2 dxdy = -I(\mu - \mu_K),$$

which yields $I(\mu - \mu_K) \le 0$. □

*Proof of* (7.8). Write $z - z_1 = re^{i\theta}$, $a = z_2 - z_1$, we have

$$\int_{|z-z_1|<R} \frac{dxdy}{|z-z_1||z-z_2|} = \int_0^R \int_0^{2\pi} \frac{drd\theta}{|re^{i\theta}-a|}.$$

Replace $\theta$ by $\theta + \alpha$, where $\alpha := \arg a$, and use $r = |a|s$, we get

$$\int_0^R \int_0^{2\pi} \frac{drd\theta}{|re^{i\theta}-a|} = \int_0^R \int_0^{2\pi} \frac{drd\theta}{|re^{i\theta}-|a||} = \int_0^{R/|a|} \int_0^{2\pi} \frac{dsd\theta}{|se^{i\theta}-1|} = \int_0^{R/|a|} \int_0^{2\pi} \frac{dsd\theta}{|s-e^{i\theta}|}.$$

Put

$$F(s) := \int_0^{2\pi} \frac{d\theta}{|s-e^{i\theta}|} = \int_0^{2\pi} \frac{d\theta}{\sqrt{s^2 - 2s\cos\theta + 1}}.$$

By Exercise 3 below, we have

$$\lim_{1>s\to 1} \frac{F(s)}{-\log(1-s)} = 2. \tag{7.9}$$

In particular, we know that $F(s)$ is integrable for $s$ near $1$. Thus

$$\int_0^T \int_0^{2\pi} \frac{dsd\theta}{|s-e^{i\theta}|} - \int_1^T \int_0^{2\pi} \frac{dsd\theta}{s} \text{ converges to a constant } C \text{ as } T \to \infty.$$

Hence $\int_1^T \int_0^{2\pi} \frac{dsd\theta}{s} = 2\pi \log T$ gives

$$\int_0^{R/|a|} \int_0^{2\pi} \frac{dsd\theta}{|s-e^{i\theta}|} = 2\pi \log \frac{R}{|a|} + C + \delta(R/|a|),$$

where $\delta(t) \to 0$ as $t \to \infty$. Now it suffices to show that

$$J(z_1, z_2, R) := \int_{|z|<R} \frac{dxdy}{|z-z_1||z-z_2|} - \int_{|z-z_1|<R} \frac{dxdy}{|z-z_1||z-z_2|}$$

goes to zero for for $R \to \infty$, uniformly when $z_1, z_2$ are on a compact set. But this follows directly from (Hint: compare with the integral of $1/s$)

$$|J(z_1, z_2, R)| \leq \int_{R-|z_1|<|z-z_1|<R+|z_1|} \frac{dxdy}{|z-z_1||z-z_2|} = \int_{\frac{R-|z_1|}{|z_1-z_2|}}^{\frac{R+|z_1|}{|z_1-z_2|}} \int_0^{2\pi} \frac{dsd\theta}{|s-e^{i\theta}|}.$$

The proof of (7.8) is now complete. □

**Proposition 13.** *In case $K = \{|z-z_0| = r\}$ or*

$$K = \{|z-z_0| \leq r\} \setminus U, \quad r > 0,$$

*where $U$ is an open set in $|z-z_0| \leq r$, we have*

$$\gamma = \log r, \quad p_{\mu_K}(z) = \max\{\log r, \log|z|\}, \tag{7.10}$$

*and $\mu_K = \frac{|dz|}{2\pi r}$ (we call it the normalized Haar measure the circle $|z-z_0| = r$).*

*Proof.* Follows directly from

$$G_{\Omega_K}(z) = G_{\{|z-z_0|>r\}}(z) = -\log|\frac{z-z_0}{r}|$$

and Lemma 2. $\square$

*Reading task*: Ransford book (subharmoic functions and potentials)

*Exercise 1*: Show that a bounded subharmonic function on $\mathbb{C}$ is a constant. *Hint: let $v$ be a bounded subharmonic function on $\mathbb{C}$ and put*

$$u(z) = \begin{cases} v(1/z) & z \neq 0 \\ \sup_{\mathbb{C}} v & z = 0. \end{cases}$$

*Show that $u$ is the upper semi-continuous regularization of* $\lim_{\varepsilon\to 0+} u_\varepsilon$, *where*

$$u_\varepsilon(z) := u(z) + \varepsilon \log|z|.$$

*Verify that $u$ is subharmonic on $\mathbb{C}$ with $u(0) = \sup_{\mathbb{C}} u$ and show it follows that $u$ is a constant.*

*Solution*: Recall that the upper semi-continuous regularization (uscr) of a function $f$ is defined by

$$f^*(z_0) := \limsup_{z\to z_0} f(z).$$

It is clear that

$$\lim_{\varepsilon\to 0+} u_\varepsilon = \begin{cases} u(z) & z \neq 0 \\ -\infty & z = 0. \end{cases}$$

To show that its uscr equals $u$, it suffices to verify that

$$\sup_{\mathbb{C}} v = \limsup_{z\to 0} u(z),$$

i.e.

$$\sup_{\mathbb{C}} v = \limsup_{z\to \infty} v(z),$$

which follows directly from the maximal principle

$$\sup_{|z|=R} v = \sup_{|z|\leq R} v, \quad R > 0,$$

for subharmonic function $v$. It is easy to see that $u(0) = \sup_{\mathbb{C}} u$. Thus it suffices to show that $u$ is subharmonic (then the maximum principle would imply that $u$ is a constant). Since $u$ is already usc and subharmonic outside $0$, it suffices to check that $u$ satisfies the submean inequality around $0$. It is clear that each $u_\varepsilon$ is subharmonic, thus its increasing limit, say

$$f := \lim_{\varepsilon\to 0+} u_\varepsilon,$$

also satisfies the submean inequality. Consider $f + C$ instead, one may assume that $f \geq 0$, then the Fatou theorem gives

$$\liminf_{z\to 0} \frac{1}{2\pi}\int_0^{2\pi} -f(z+re^{i\theta})\,d\theta \geq \frac{1}{2\pi}\int_0^{2\pi} \liminf_{z\to 0}\{-f(z+re^{i\theta})\}\,d\theta,$$

which implies

$$u(0) = \limsup_{z\to 0} f(z) \le \limsup_{z\to 0} \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})\,d\theta \le \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})\,d\theta.$$

Thus $u$ satisfies the submean inequality around $0$. The proof is complete. The above proof actually only assume that $\sup_{\mathbb{C}} v < \infty$. Hence we obtain

**Theorem 51.** *Let $v$ ve a subharmonic function on $\mathbb{C}$. If $\sup_{\mathbb{C}} v < \infty$ then $v$ is a constant.*

*Exercise 2*: Verify the following envelope formula of the equilibrium potential $p_{\mu_K}$:

$$p_{\mu_K} = G_K + \gamma,$$

where $\gamma = \lim_{z\to\infty} \log|z| - G_K(z)$ is the Robin constant and

$$G_K := \sup\{v : v \text{ is subharmonic on } \mathbb{C},\, v\le 0 \text{ on } K \text{ and } \limsup_{z\to\infty} v(z) - \log|z| < \infty\}.$$

*Solution*: Notice that both $G_K$ and $p_{\mu_K}-\gamma$ are zero on $\partial\Omega_K$, harmonic on $\Omega_K$ and satisfies that the difference to $\log|z|$ is bounded at $\infty$. Thus the maximum principle (try!) gives $G_K = p_{\mu_K}-\gamma$ on $\overline{\Omega_K}$. On the other hand, since $p_{\mu_K}-\gamma$ is a candidate for $G_K$, we also have $p_{\mu_K}-\gamma \le G_K$. but the maximum principle directly gives $G_K \le 0 = p_{\mu_K}-\gamma$ on $\mathbb{C}\setminus\Omega_K$, thus we must have $p_{\mu_K}-\gamma = G_K$ on $\mathbb{C}\setminus\Omega_K$. The proof is complete.

*Exercise 3*: Prove (7.9). *Hint: use the taylor expansion of $\cos\theta$ around $\theta = 0$.*

*Solution*: To see this, one may use that $\cos\theta = 1-\theta^2/2 + o(|\theta|^2)$ near $\theta = 0$ and check that

$$\lim_{s\to 1} \frac{\int_0^1 \frac{d\theta}{\sqrt{s^2-2s(1-\theta^2/2)+1}}}{-\log(1-s)} = 1.$$

In fact, by a change of variable $\theta = t(1-s)/\sqrt{s}$, we have

$$\int_0^1 \frac{d\theta}{\sqrt{s^2-2s(1-\theta^2/2)+1}} = \frac{1}{\sqrt{s}}\int_0^{\frac{\sqrt{s}}{1-s}} \frac{dt}{\sqrt{t^2+1}},$$

which can be compared with $\int_1^{\frac{1}{1-s}} \frac{dt}{t} = -\log(1-s)$ as $s\to 1-$.

7.4. **Robin constant, capacity and transfinite diameter.** We shall follow page 23-24 in [A1] and page 153-154 (Fekete–Szegö theorem) in [Ra] to prove that $\gamma = \log c(K)$.

**Definition 35.** *Let $K$ be a compact set in $\mathbb{C}$. The **order $n$ diameter** ($n\ge 2$) of $K$ is defines as*

$$d_n := \sup\left\{ \prod_{1\le j<k\le n} |z_j - z_k|^{\frac{2}{n(n-1)}} : z_1,\cdots,z_n \in K \right\}.$$

*An $n$-tuple $z_1,\cdots,z_n\in K$ where the supremum is attained is called a **Fekete $n$-tuple** for $K$.*

It is easy to see that $d_2$ is the diameter of $K$.

**Lemma 4.** *$d_{n+1} \le d_n$ for all $n\ge 2$.*

*Proof.* Note that for for each $1 \le l \le n+1$, we have

$$\prod_{1\le j<k\le n+1;\, j,k\neq l} |z_j - z_k|^{\frac{2}{n(n-1)}} \le d_n.$$

When $l$ goes through $1, \cdots, n+1$, each $|z_j - z_k|$ occurs $n-1$ times, hence

$$\prod_{1\le j<k\le n+1} |z_j - z_k|^{\frac{2}{n}} = \prod_{1\le l\le n} \prod_{1\le j<k\le n+1;\, j,k\neq l} |z_j - z_k|^{\frac{2}{n(n-1)}} \le d_n^{n+1}$$

gives $d_{n+1} \le d_n$. □

**Definition 36.** *We call*

$$d_\infty := \lim_{n\to\infty} d_\infty$$

*the* ***transfinite diameter*** *of $K$.*

**Theorem 52** (Fekete-Szegö theorem)**.** *Let $K$ be a compact set in $\mathbb{C}$. Assume that $\mathbb{C} \setminus K$ has real analytic boundary. Then*

$$\gamma = \log d_\infty. \tag{7.11}$$

*Proof. Step 1: $\gamma \le \log d_\infty$.* Let $z_1, \cdots, z_n \in K$ be the Fekete $n$-tuple, then the Fekete polynomial $F_n$ defined by

$$F_n(z) := (z - z_1) \cdots (z - z_n)$$

satisfies

$$p_\mu = \frac{1}{n} \log |F_n|, \quad \mu := \frac{\delta_{z_1} + \cdots + \delta_{z_n}}{n}.$$

Note that if $z \in K$ then

$$\prod_{i=1}^{n} |z - z_j| \prod_{j<k} |z_j - z_k| \le (\delta_{n+1})^{\frac{n(n+1)}{2}}$$

gives

$$|F_n(z)| \le \frac{(\delta_{n+1})^{\frac{n(n+1)}{2}}}{(\delta_n)^{\frac{n(n-1)}{2}}} \le \frac{(\delta_n)^{\frac{n(n+1)}{2}}}{(\delta_n)^{\frac{n(n-1)}{2}}} = (\delta_n)^n,$$

i.e.

$$\gamma \le \sup_K p_\mu \le \log \delta_n.$$

Hence it suffices to let $n \to \infty$.

*Step 2: $\gamma \ge \log d_\infty$.* Still let $z_1, \cdots, z_n \in K$ be the Fekete $n$-tuple so that

$$\prod_{j<k} |z_j - z_k| = (\delta_n)^{\frac{n(n-1)}{2}}.$$

For each $1 \le j \le n$, let $\mu_j$ be the normalized Haar measure on the circle $|z - z_j| = \varepsilon$, and put

$$\mu = \frac{\mu_1 + \cdots + \mu_n}{n},$$

we have

$$I(\mu)=\frac{1}{n^2}\sum_{j=1}^{n} I(\mu_j)+\frac{2}{n^2}\sum_{j<k} I(\mu_j,\mu_k).$$

Recall that $I(\mu_j)=\log\varepsilon$ by Proposition 13 and the submean inequality gives

$$I(\mu_j,\mu_k)=\int_{|w-z_k|=\varepsilon} p_{\mu_j}(w)d\mu_k(w)\geq p_{\mu_j}(z_k)=\int_{|z-z_j|=\varepsilon}\log|z-z_k|d\mu_j(z)\geq \log|z_j-z_k|.$$

Hence

$$I(\mu)\geq \frac{\log\varepsilon}{n}+\frac{n-1}{n}\log\delta_n.$$

Since $\mu$ is supported on

$$K_\varepsilon:=\{z\in\mathbb{C}:\mathrm{dist}\,(z,K)\leq\varepsilon,\}$$

let $n\to\infty$, we know that the Robin constant $\gamma_\varepsilon$ of $K_\varepsilon$ satisfies $\gamma_\varepsilon\geq\log d_\infty$. Let $\varepsilon\to 0$, we finally get (try!) $\gamma\geq\log d_\infty$. $\square$

**Remark.** (7.11) *is true for general compact set* $K\subset\mathbb{C}$, *see* [Ra, page 153-154] *for the proof.*

**Theorem 53.** *Let $K$ be a compact set in $\mathbb{C}$. Then*

$$c(K)=d_\infty, \tag{7.12}$$

*where $c(K)$ is the capacity of $K$ defined in Definition 28.*

*Proof.* Put

$$\rho_n=\inf_{a_1,\cdots,a_n\in\mathbb{C}}\max_{z\in K}|z^n+a_1z^{n-1}+\cdots+a_n|^{1/n}.$$

We know that $\inf_{n\geq 1}\rho_n=c(K)$. The proof consists of three steps.

*Step 1:* $\rho_n\leq d_n$. Since $K$ is compact, we can choose $z_1,\cdots,z_n\in K$ such that

$$d_n=\prod_{1\leq j<k\leq n}|z_j-z_k|^{\frac{2}{n(n-1)}}.$$

Consider

$$V(z_{n+1}):=\prod_{1\leq j<k\leq n+1}|z_j-z_k|\leq (d_{n+1})^{\frac{n(n+1)}{2}}\leq (d_n)^{\frac{n(n+1)}{2}}.$$

Write $z_{n+1}=z$ and think of $V$ as a polynomial of $z\in K$. The leading coefficient $a_n$ of $V$ satisfies

$$|a_n|=(d_n)^{\frac{n(n-1)}{2}}.$$

Thus

$$\rho_n\leq\left(\sup_K\left|\frac{V}{a_n}\right|\right)^{\frac{1}{n}}\leq\frac{(d_n)^{\frac{n+1}{2}}}{(d_n)^{\frac{n-1}{2}}}=d_n.$$

as a function of $z$ By the definition of $d_{n+1}$, we have

*Step 2:* $\limsup_{n\to\infty}\rho_n \le c(K) \le \liminf_{n\to\infty}\rho_n$. Since $K$ is compact, there exists (try!) a degree $n$ monic polynomial $T_n$ such that $\sup_K |T_n| = \rho_n^n$. Note that (by the extremal property of $T_{mk+h}$)

$$\sup_K |T_{mk+h}| \le \sup_K |T_m^k T_h| \le \left(\sup_K |T_m|\right)^k \sup_k |T_h|$$

gives

$$(\rho_{mk+h})^{mk+h} \le (\rho_m)^{mk} (\rho_h)^h,$$

i.e.

$$\log \rho_{mk+h} \le \frac{mk}{mk+h}\log\rho_m + \frac{h}{mk+h}\log\rho_h.$$

Fix $m \ge 1$, let $h$ run through $0, \cdots, m-1$ and $k\to\infty$, the above inequality gives

$$\limsup_{n\to\infty} \log\rho_n \le \rho_m.$$

Thus step 2 follows.

*Step 3:* $d_\infty \le c(K)$. Recall the following identity for the Vandermonde matrix

$$\prod_{1\le j<k\le n}(z_j - z_k) = \det V, \quad V := \det\begin{pmatrix} z_1^{n-1} & z_1^{n-2} & \cdots & z_1 & 1 \\ z_2^{n-1} & z_2^{n-2} & \cdots & z_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ z_n^{n-1} & z_n^{n-2} & \cdots & z_n & 1. \end{pmatrix}$$

Let $P_n$ be arbitrary degree $n$ monic polynomials, apply the column transform, we have

$$\prod_{1\le j<k\le n}(z_j - z_k) = \det V = \det\begin{pmatrix} P_{n-1}(z_1) & P_{n-2}(z_1) & \cdots & P_1(z_1) & 1 \\ P_{n-1}(z_2) & P_{n-2}(z_2) & \cdots & P_1(z_2) & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{n-1}(z_n) & P_{n-2}(z_n) & \cdots & P_1(z_n) & 1 \end{pmatrix},$$

which implies that

$$(d_n)^{\frac{n(n-1)}{2}} \le n! \prod_{j=1}^{n-1} \sup_K |P_j|.$$

Take the infimum over all $P_j$, we get

$$(d_n)^{\frac{n(n-1)}{2}} \le n! \prod_{j=1}^{n-1} (\rho_j)^j,$$

i.e.

$$\log d_n \le \frac{\log\rho_1 + \cdots + (n-1)\log\rho_{n-1}}{1+\cdots+(n-1)}.$$

Let $n\to\infty$ ($\lim_{n\to\infty}\log\rho_n = \log c(K)$ by step 2), we get

$$\log d_\infty \le \log c(K).$$

Thus step 3 follows. Together with step 1, we get $c(K) = d_\infty$. (16th Apr) $\square$

7.4.1. *A short summary.*

1. The first main result is the Cauchy integral theorem (see Theorem 1), which implies Theorem 4, Theorem 9 and Theorem 13, etc.

For example, Theorem 1) directly gives (try!)

$$\int_{|z|=\pi} z^{20} + \sin z \, dz = 0$$

and can be used to prove Corollary 1, which is used in the proof of the Riemann mapping theorem. Theorem 9 can be used to compute integrals (see page 154-161 in the Ahlfors book, especially the exercise in page 161)

$$\int_0^{2\pi} \frac{dt}{a + b\cos t} = \frac{2\pi}{\sqrt{a^2 - b^2}}, \quad a > b \geq 0; \quad \int_{-\infty}^{\infty} \frac{e^{-ix}\, dx}{1 + x^2} = \frac{\pi}{e}$$

and

$$\int_{|z|=1} \frac{|dz|}{|z-3|^2} = \frac{\pi}{4}.$$

Theorem 13 can be used to prove that

$$z + e^{-z} = \lambda, \quad \lambda > 1 \text{ is a constant},$$

has exactly one solution in the right half plane and prove Corollary 4, Corollary 5, Theorem 15 and Theorem 16 etc.

2. The second main result is the Riemann mapping theorem, Theorem 19. We know that the Riemann mapping function is directly related to the Green function of a simply connected domain (see Proposition 8). In order to study the regularity property of the Riemann mapping function (see Theorem 42), we introduce the theory of harmonic functions. We prove the mean value property, Theorem 26, for harmonic functions, and obtain the Poisson formula, Theorem 28, using the mean value property and the Mobius transform. Then we prove the crucial Schwarz's theroem, Theorem 29, for the Poisson integral and the Harnack inequality, Theorem 31, for positive harmonic functions. Applications include the Harnack principle — Theorem 32 (which is used in the proof of Theorem 35) and the reflection principle —Theorem 39 (which implies Theorem 42 — the regularity property of the Riemann mapping function).

3. The third main result is the solution of the *Dirichlet Problem*, Theorem 36, which is used to define the crucial Green's function for a regular domain. We use the reflection principle to prove Theorem 41 — the regularity property of Dirichlet Problem. Then we use Green's function to define the Poisson kernel, harmonic measure, Robin constant, equilibrium measure and the equilibrium potential.

4. The final part is on the potential theory. The main result is the extremal property, Theorem 50, of equilibrium measure, which is a direct consequence of a Green type formula — (5.12).

7.4.2. *Test exam 2 (NOT a normal exam, just a collection of related exercises).*

**Exercise 1.** Show that
$$\int_0^{2\pi} \log|e^{i\theta}+3|\,d\theta = 2\pi\,\log 3$$
and
$$\log 2 \leq \frac{1}{2\pi i}\int_{|z|=20} \log|z^9+8z^7+2|\,\frac{dz}{z}.$$

*Answer:* Note that $|z+3|>0$ on the disk $|z|\leq 1$, we know that the holomorphic function $z+3$ has no zero in $|z|\leq 1$, thus $\log|z+8|$ is harmonic on $|z|\leq 1$ and the first identity follows from the mean-value property. The second inequality follows from the submean inequality for the subharmonic function $\log|z^9+8z^7+2|$ (note also that $dz/z = i\theta$ for $z=20e^{i\theta}$).

**Exercise 2.** Let $u$ be a positive subharmonic function on a domain $\Omega\subset\mathbb{C}$. Assume that the disk $|z|\leq\rho$ lies in $\Omega$. Show that
$$u(z)\leq \frac{1}{2\pi}\,\frac{\rho+r}{\rho-r}\int_0^{2\pi} u(\rho e^{i\theta})\,d\theta$$
for all $z$ with $|z|=r<\rho$. You might use the maximum principle, Schwarz's theorem and the proof of the Harnack inequality.

*Answer:* Since subharmonic function is the decreasing limit of smooth subharmonic function (see Exercise 4 below), one may assume that $u$ is continuous. By Schwartz's theorem, one can take a harmonic function $v$ on the disk $|z|<\rho$ such that $v$ is continuous on $|z|\leq\rho$ and $v=u$ on $|z|=\rho$, then the maximum principle for the subharmonic function $u-v$ implies that $u\leq v$ on $|z|\leq\rho$, thus Poisson's formula (3.19) gives
$$u(z)\leq v(z) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\rho^2-|z|^2}{|\rho e^{i\theta}-z|^2}\,v(\rho e^{i\theta})\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\frac{\rho^2-|z|^2}{|\rho e^{i\theta}-z|^2}\,u(\rho e^{i\theta})\,d\theta$$
for all $z$ with $|z|<\rho$. If $|z|=r<\rho$, then
$$\frac{\rho^2-|z|^2}{|\rho e^{i\theta}-z|^2}\leq\frac{\rho^2-r^2}{(\rho-r)^2}=\frac{\rho+r}{\rho-r}$$
gives the estimate that we need. The second proof is to use (assume that $u$ is smooth)
$$u(z)=\frac{1}{2\pi}\int_{|w|<\rho}\log\left|\frac{\rho(z-w)}{\rho^2-z\bar w}\right|\Delta u(w)\,dxdy+\frac{1}{2\pi}\int_0^{2\pi}\frac{\rho^2-|z|^2}{|\rho e^{i\theta}-z|^2}\,u(\rho e^{i\theta})\,d\theta.$$
Then $\Delta u\geq 0$ and $\log|\frac{\rho(z-w)}{\rho^2-z\bar w}|\leq 0$ gives
$$u(z)\leq\frac{1}{2\pi}\int_0^{2\pi}\frac{\rho^2-|z|^2}{|\rho e^{i\theta}-z|^2}\,u(\rho e^{i\theta})\,d\theta,$$
the remaining steps are the same.

**Exercise 3(optional).** Show that the domain
$$\Omega_1 := \{x+iy\in\mathbb{C} : x^2+y^2<1,\ y>|x|\}$$

is regular, but $\Omega_2 := \Omega_1 \setminus \{i/2\}$ is not regular.

*Answer:* One may check that every boundary point of $\Omega_1$ possesses a barrier, for example, $\omega(x+iy) = y$ defines a barrier of $\Omega_1$ at the origin (or we can directly use the fact that every bounded domain with continuous boundary is regular). One the other hand, $\Omega_2$ is not regular since it does not possess a barrier at $i/2$ (otherwise we would have a positive harmonic function $u$ on $\Omega_2$ such that $u$ extends to a continuous function on the closure of $\Omega_2$ with $u(i/2) = 0$. Then (see Exercise 1 in page 45) we know that $u$ is in fact harmonic on $\Omega_1$ with minimum point at $i/2$, which contradicts with the maximum principle.

**Exercise 3.** Let $-\infty \le a < b \le \infty$, let $u : \Omega \to (a,b)$ be a harmonic function on an open set $\Omega \subset \mathbb{C}$, and let $\chi : (a,b) \to \mathbb{R}$ be a convex function. Show that $\chi \circ u$ is subharmonic on $\Omega$.

*Answer:* By taking a convolution with an even function, we know that $\chi$ is the decreasing limit of smooth convex functions. Thus we can assume that $\chi$ is smooth and convex. Then we have

$$\chi(u)_z = \chi'(u)u_z, \ \chi(u)_{z\bar z} = \chi''(u)|u_z|^2 + \chi'(u)u_{z\bar z}.$$

Since $u$ is harmonic we have $u_{z\bar z} = 0$, thus

$$\chi(u)_{z\bar z} = \chi''(u)|u_z|^2 \ge 0,$$

(note that convexity of $\chi$ gives $\chi'' \ge 0$) gives $\Delta\chi \circ u \ge 0$. Now we can just use the fact that a smooth function $v$ is subharmonic if and only if $\Delta v \ge 0$ — follows from Green's formula (try)

$$v(a) = \frac{1}{2\pi}\int_{|z-a|<r} \log|(z-a)/r|\, \Delta v\, dxdy + \frac{1}{2\pi}\int_0^{2\pi} v(a+re^{i\theta})\, d\theta$$

for $v$ and $G(z) := \log|(z-a)/r|$ on the disk $|z-a| < r$.

**Exercise 4.** Let $u$ be a subharmonic function on an open set $\Omega$ in $\mathbb{C}$. Let $\chi : \mathbb{C} \to [0,\infty)$ be a smooth function with $\chi(w) = 0$ for all $|w| > 1$, $\chi(w) = \chi(|w|)$ and $\int_{\mathbb{C}} \chi(w)d\lambda_w = 1$, where $d\lambda_w$ denotes the Lebesgue measure. For $\varepsilon > 0$, put

$$\chi_\varepsilon(w) = \frac{1}{\varepsilon^2}\chi(w/\varepsilon),$$

and define the following convolution

$$u_\varepsilon(z) := \int_{|w|<\varepsilon} u(z-w)\chi_\varepsilon(w)d\lambda_w$$

for

$$z \in \Omega_\varepsilon := \{z \in \Omega : |z-w| > \varepsilon, \forall\, w \in \partial\Omega\}.$$

Show that:

a) $u_\varepsilon$ is smooth subharmonic on $\Omega_\varepsilon$;

b) $u_\varepsilon$ decreases to $u$ as $\varepsilon \to 0$;

c) $u_\varepsilon = u$ if $u$ is harmonic.

*Answer:* (a) follows from

$$u_\varepsilon(z) = \int_{\mathbb{C}} u(z-w)\chi_\varepsilon(w)d\lambda_w = \int_{\mathbb{C}} u(w)\chi_\varepsilon(z-w)d\lambda_w$$

and the fact that $\chi_\varepsilon$ is smooth. (b) follows from

$$\int_{|w|<1} u(z-\varepsilon w)\chi(w)d\lambda_w = \int_0^1 \int_0^{2\pi} u(z-\varepsilon re^{i\theta})\chi(r)\, rdrd\theta \tag{7.13}$$

(since $\int_0^{2\pi} u(z-\varepsilon re^{i\theta})d\theta$ decreases to $2\pi u(z)$ as $\varepsilon \to 0$, in fact, put

$$v(w) = \int_0^{2\pi} u(z-we^{i\theta})d\theta,$$

the Fubini theorem implies that $v$ is subharmonic with $v(w) = v(|w|)$ thus the maximum principle implies that

$$v(t) = \sup_{|w|\leq t} v(w)$$

is increasing with respect to $t$). (c) follows from (7.13) and the mean value property

$$\int_0^{2\pi} u(z-\varepsilon re^{i\theta})d\theta = 2\pi u(z)$$

for harmonic $u$.

**Exercise 5.** Let $u$ be a smooth subharmonic function $\mathbb{C}$. Put

$$C_u(z) := \frac{1}{2\pi}\int_0^{2\pi} u(e^z e^{i\theta})\, d\theta,$$

show that

(a) $C_u$ is smooth subharmonic and depends only on $\operatorname{Re} z$ in $\mathbb{C}$;

(b) $C_u(t)$ is convex increasing with respect to $t \in \mathbb{R}$;

(c) Assume further that $\sup_{\mathbb{C}} u < \infty$, show that $C_u$ is a constant.

*Answer:* (a) follows from the Fubini theorem and a change of variable $\theta' := \theta + \operatorname{Im} z$. (b) write $z = t + is$, since $C_u$ depends only on $t = \operatorname{Re} z$ we have

$$0 \leq \Delta C_u = (C_u)_{tt},$$

which implies that $C_u$ is convex and bounded at $-\infty$, thus $C_u$ is also increasing. To prove (c), note that if $\sup_{\mathbb{C}} u < \infty$ then $C_u$ is also bounded at $\infty$, thus $C_u$ is also decreasing, hence $C_u$ must be a constant.

**Exercise 6.**

(a) Compute the equilibrium potential of circle $|z-a| = r$;

(b) Show that $f(z) := z + z^{-1}$ is conformal from $|z| > 1$ onto $\mathbb{C} \setminus [-2, 2]$ and compute the Green function of $\mathbb{C} \setminus [-2, 2]$ with a pole at $\infty$;

(c) Use (b) to compute the Robin constant and capacity of $[-2, 2]$ (Hint: observe that they are equal to the Robin constant and capacity of the unit disk);

(d) Show that the capacity of $[c_1, c_2]$ is $(c_2 - c_1)/4$.

*Answer:* (a): recall that the equilibrium measure for $|z-a|=r$ is equal to $d\theta/(2\pi)$, thus its equilibrium potential is given by

$$\int_0^{2\pi} \log|z-(a+re^{i\theta})| \frac{d\theta}{2\pi} = \max\{\log|z-a|, \log r\}.$$

(b) To see that $f(z) := z+z^{-1}$ is conformal on $|z|>1$, we need to show that it is injective, i.e. if $f(z)=f(w)$ then

$$z+z^{-1} = w+w^{-1}$$

gives $z=w$. In fact, if $z \neq w$ then the above equality gives $1=1/zw$, which contradicts with $|z|, |w|>1$. To show that it is surjective, one may observe that it maps $|z|=r>1$ to a ellipse around $[-2,2]$. Now we know that $G(w) := -\log|f^{-1}(w)|$ is the Green function of $\mathbb{C}\setminus[-2,2]$ with a pole at $\infty$. For (c), we know that the Robin constant of [-2,2] is given by

$$\lim_{w\to\infty} -\log|f^{-1}(w)| + \log|w| = \lim_{z\to\infty} -\log|z| + \log|z+z^{-1}| = 0.$$

Thus the capacity of [-2,2] is $1$. For (d), we know from the definition of the capacity that $c(aK+b)=|a|c(K)$ for constant $a, b$. Thus (d) follows from (c).

**Exercise 7.** Let $f$ be a conformal mapping from a bounded simply connected domain $\Omega$ onto the unit disk such that $f(a)=0$ for some $a\in\Omega$.

a) Find Green's function of $\Omega$ with a pole at $a$;

b) Show that the Robin constant of $\mathbb{C}\setminus\Omega'$ is $\log|f'(a)|$, where

$$\Omega' := \{(z-a)^{-1} : z\in\Omega\setminus\{a\}\}.$$

*Answer:* (a): $\log|f(z)|$ (b) write $\zeta = 1/(z-a)$, we get $z=1/\zeta+a$. Thus the Green's function for $\Omega'$ with a pole at $\infty$ is given by $-\log|f(1/\zeta+a)|$. Hence its Robin constant is given by

$$\lim_{\zeta\to\infty} \log|f(1/\zeta+a)| + \log|\zeta| = \log|f'(a)|.$$

**Exercise 8.** Let $\mu$ be a Borel probability measure on a compact set $K\subset\mathbb{C}$. Assume that $p_\mu$ is a constant $c$ on $K$. Show that

$$c = \sup_\nu \inf_{z\in\mathbb{C}} p_\nu(z) = \inf_\nu \sup_{z\in K} p_\nu(z),$$

where $\nu$ is taken over the space of Borel probability measure on $K$.

*Answer:* see *Step 1* in the proof of Theorem 50.

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DEPARTMENT OF MATHEMATICAL SCIENCES, NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY, NO-7491 TRONDHEIM, NORWAY

*Email address*: `xu.wang@ntnu.no`