SOLUTIONS

FOR PROBLEMS:

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PROBLEM 2:

We want to reflect the Imaginary axis $z = iy$, the line $x = y$ and the circle $|z| = 1$ in the circle $|z-2| = 1$

The formula for the reflection is

$$z^* = \frac{R^2}{\bar{z} - \bar{a}} + a$$

In our case $a = 2$ and $R = 1$

so

$$z^* = \frac{1}{\bar{z} - 2} + 2 = \frac{2\bar{z} - 3}{\bar{z} - 2} = \overline{\left(\frac{2z-3}{z-2}\right)}$$

$T(z) = \frac{2z-3}{z-2}$ will map lines and circles to lines and circles and $\overline{T(z)}$ will do the same

This means that for each of the set $z = iy$, $x = y$ and $|z| = 1$ we only need to look for the image of 3 points and then find the circle containing these 3 images.

1) $z = iy$

Here we can look for the image of ∞, 0 and i

$$T(0) = \frac{3}{2}$$

$$T(\infty) = 2$$

$$T(i) = \frac{-2i-3}{-i-2} = \frac{2i+3}{i+2} = \frac{(2i+3)(-i+2)}{1+4}$$

$$= \frac{2-3i+4i+6}{5} = \frac{8}{5} + \frac{1}{5}i$$

Now we need to find the center and the radius of the circle containing $\frac{3}{2}$, 2 and $\frac{8}{5}+\frac{1}{5}i$

The center $\underline{a+ib}$ will have the property that:

$$\left|\frac{3}{2} - a - ib\right|^2 = |2-a-ib|^2$$

$$= \left|\frac{8}{5} + \frac{1}{5}i - a - ib\right|^2$$

③

$$\left|\frac{3}{2}-a+ib\right|^2 = |2-a+ib|^2$$

$$\Downarrow$$

$$\left(\frac{3}{2}-a\right)^2 \pm (2-a)^2$$

$$\frac{9}{4}-3a+a^2 = 4-4a+a^2$$

$$\Downarrow$$

$$\frac{9}{4}-3a = 4-4a$$

$$a = 4-\frac{9}{4} = \frac{7}{4}$$

To find b we use.

$$|2-a-ib|^2 = \left|\frac{8}{5}+\frac{1}{5}i-a-ib\right|$$

$$a = \frac{7}{4}$$

$$\Downarrow$$

$$\left|2-\frac{7}{4}-ib\right|^2 = \left|\frac{8}{5}-\frac{7}{4}+i\left(\frac{1}{5}-b\right)\right|^2$$

$$\Downarrow$$

$$\left|\frac{1}{4}+ib\right|^2 = \left|-\frac{3}{20}+i\left(\frac{1}{5}-b\right)\right|^2$$

$$\Downarrow$$

$$\frac{1}{16}+b^2 = \left(-\frac{3}{20}\right)^2+\frac{1}{25}-\frac{2}{5}b+b^2$$

$$\Downarrow$$

$$\frac{1}{16} = \left(-\frac{3}{20}\right)^2+\frac{1}{25}-\frac{2}{5}b$$

$$\Downarrow$$

$$\frac{5}{16} = \frac{9}{80}+\frac{1}{25}-2b$$

$$\underline{\underline{b = -\frac{2}{25}}}$$

④

The radius of the circle is R

$$R^2 = |2 - a - ib|^2$$

$$= |2 - \frac{7}{4} + i\frac{2}{25}|^2 =$$

$$|\frac{1}{4} + i\frac{2}{25}|^2$$

$$= \frac{1}{16} + \frac{4}{(25)^2}$$

image of the line

~~ix~~ $z = iy$ is

$$\underline{|z - \frac{7}{4} + i\frac{2}{25}|^2 = \frac{1}{16} + \frac{4}{(25)^2}}$$

Now the line $x = y$

Look at the image of

3 points on this line

$0, 1+i, \infty$ is on the line

$$T(0) = \frac{3}{2}$$

$$T(\infty) = 2$$

$$T(1+i) = \frac{2 - 2i - 3}{1 - i - 2} = \frac{-2i - 1}{-i - 1}$$

$$= \frac{2i+1}{i+1} = \frac{3}{2} + \frac{1}{2}i$$

The center of the circle

$a + i b$.

$$|2 - a - ib|^2 = \left|\frac{3}{2} - a - ib\right|^2 = \left|\frac{3}{2} + \frac{1}{2}i - a - ib\right|^2$$

$$|2 - a - ib|^2 = \left|\frac{3}{2} - a - ib\right|^2$$

$$\Downarrow$$

$$(2-a)^2 + b^2 = \left(\frac{3}{2} - a\right)^2 + b^2$$

$$\Downarrow$$

$$(2-a)^2 = \left(\frac{3}{2} - a\right)^2$$

$$\Downarrow$$

$$\underline{a = \frac{7}{4}}$$

To get b

$$\left|\frac{3}{2} - a - ib\right|^2 = \left|\frac{3}{2} - a + i\left(\frac{1}{2} - b\right)\right|^2$$

$$\Downarrow$$

$$\left|\frac{1}{4} - ib\right|^2 = \left|-\frac{1}{4} + i\left(\frac{1}{2} - b\right)\right|^2$$

$$\Downarrow$$

$$\frac{1}{16} + b^2 = \frac{1}{16} + \frac{1}{4} - b + b^2$$

$$b = \frac{1}{4}$$

Center is $\frac{7}{4} + \frac{1}{4}i$

The radius

$$R^2 = \left|\frac{1}{4} - i\frac{1}{4}\right|^2 = \frac{1}{16} + \frac{1}{16} = \frac{2}{16}$$

$$R = \frac{\sqrt{2}}{4}$$

⑥

Finally the circle $|z|=1$

The points $-1, i, 1$ are on the circle.

$$\overline{T}(-1) = \frac{-2-3}{-1-2} = \frac{5}{3}$$

$$\overline{T}(i) = \frac{-2i-3}{-i-2} = \frac{2i+3}{i+2} = \frac{8}{5} + \frac{1}{5}i$$

$$\overline{T}(1) = \frac{2-3}{1-2} = 1$$

Center

$a+ib$

$$\left|\frac{5}{3} - a - ib\right|^2 = |1-a-ib|^2$$

$$= \left|\frac{8}{5} + \frac{1}{5}i - a - ib\right|^2$$

$$\Downarrow$$

$$\left(\frac{5}{3}-a\right)^2 = (1-a)^2$$

$$\frac{25}{9} - \frac{10}{3}a = 1 - 2a$$

$$a = \frac{4}{3}$$

$$\left|1 - \frac{4}{3} + ib\right|^2 = \left|\frac{8}{5} - \frac{4}{3} + i\left(\frac{1}{5} - b\right)\right|^2$$

$$\frac{1}{9} + b^2 = \left(\frac{4}{15}\right)^2 + \frac{1}{25} - \frac{2}{5}b + b^2$$

$$\underline{b = 0}$$

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The center is $\frac{4}{3}$

the radius

$$R^2 = \left|1 - \frac{4}{3}\right|^2 = \frac{1}{9}$$

$$\underline{R = \frac{1}{3}}$$

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Problem 3

Compute

$$\int_{|z|=2} \frac{dz}{z^2-1}$$

$|z|=2$ traveled in the positive direction.

$$\frac{1}{2}\int_{|z|=2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)dz$$

$$= \frac{1}{2}\int_{|z|=2}\frac{1}{z-1}\,dz - \frac{1}{2}\int_{|z|=2}\frac{1}{z+1}\,dz$$

Let us look at

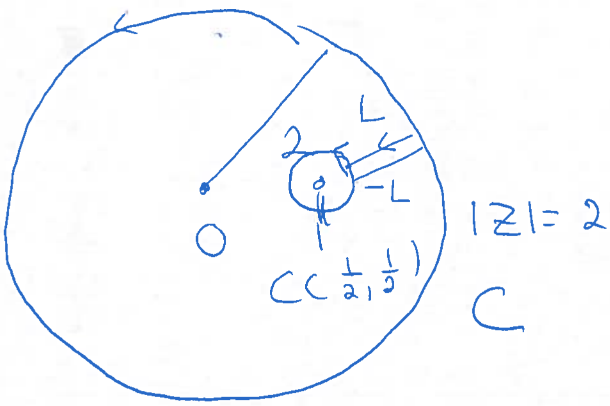
$$\int_{|z|=2}\frac{1}{z-1}\,dz$$ and

$$\int_{|z|}\frac{1}{z+1}\,dz$$

Separately we can use a standard trick.

$$\int_{|z|=2} \frac{1}{z-1}\,dz$$

Let $C = C(1, \tfrac{1}{2}) = \{z : |z-1| = \tfrac{1}{2}\}$



Connect $|z|=2$ to $C(1, \tfrac{1}{2})$

with a straight line L

Then

$$T_1 = C \cup L \cup -L \cup (-C(\cancel{1}, \tfrac{1}{2}))$$

is closed and do not

surround $\cancel{0}$

So

$$\int_{\Gamma} \frac{1}{z-1}\,dz = 0$$

$$\int_{\Gamma} \frac{1}{z-1}\,dz = \int_{C} \frac{1}{z-1}\,dz + \int_{L} \frac{1}{z-1}\,dz + \int_{-L} \frac{1}{z-1}\,dz$$

$$+ \int_{-C(1,\frac{1}{2})} \frac{1}{z-1}\,dz =$$

$$\int_{C} \frac{1}{z-1}\,dz - \int_{C(1,\frac{1}{2})} \frac{1}{z-1}\,dz$$

Now $\displaystyle\int_{C(1,\frac{1}{2})} \frac{1}{z-1}\,dz = 2\pi i$ so

$$\frac{1}{2}\int_{C} \frac{1}{z-1}\,dz = \pi i$$

Use the same idea for.

$$\frac{1}{2}\int_C \frac{1}{z+1}\,dz$$ and find that

$$\frac{1}{2}\int_C \frac{1}{z+1}\,dz = \frac{1}{2}\int_{C(-1,\frac{1}{2})} \frac{1}{z+1}\,dz = \frac{1}{2}2\pi i = \pi i$$

where $C(-1,\frac{1}{2})$ is the curve.

$|z+1| = \frac{1}{2}$ So

$$\int_C \frac{1}{z^2-1}\,dz = \frac{1}{2}\int_C \frac{1}{z-1}\,dz - \frac{1}{2}\int_C \frac{1}{z+1}\,dz$$

$$= \pi i - \pi i = 0.$$

Problem 6

f analytic in Ω and

$|f(z)-1| < 1$ in Ω.

If $|f(z)-1| < 1$ it follows that

$f(z) \neq 0$ in Ω which means

that $\log f(z)$ is well defined

in Ω

If γ is a closed curve

in Ω then γ is parametrized by

$z(t)$, $a \leq t \leq b$ and $z(a) = z(b)$

So $$\int_\gamma \frac{f'(z)}{f(z)}\, dz = \int_a^b \frac{f'(z(t))}{f(z(t))}\, z'(t)\, dt$$

$$= \log f(z(b)) - \log f(z(a)) = 0$$

Problem 8:

Let $f(z) = \log z$,

then f is analytic in

$\mathbb{C} \setminus L$ where L is a ray from

0 to infinity.

Further $f'(z) = \frac{1}{z}$ which is continuous

in $\mathbb{C} \setminus L$. So if $\gamma \subset \mathbb{C} \setminus L$ and

γ is closed, then.

$$\int_\gamma f(z)\,dz = \int_\gamma \log z \, dz = 0$$

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Problem 1:

$$\int_{|z|=1} e^{z} z^{-n} dz = \int_{|z|=1} \frac{e^{z}}{z^{n}} dz$$

this makes me think of the derivative of e^z $n-1$ times at 0

$$f(z) = e^{z}$$

$$f^{(n-1)}(0) = \frac{(n-1)!}{2\pi i} \int_{|z|=1} \frac{f(z)}{z^{n}} dz$$

$$f^{(n-1)}(z) = e^{z}$$ so

$$f^{(n-1)}(0) = 1$$ hence

$$\frac{(n-1)!}{2\pi i} \int_{|z|=1} \frac{e^{z}}{z^{n}} dz = 1$$

$$\Downarrow$$

$$\int_{|z|=1} \frac{e^{z}}{z^{n}} dz = \frac{2\pi i}{(n-1)!}$$

$$\int_{|z|=2} z^n (1-z)^m \, dz = 0 \quad \text{if} \quad n, m \geq 1$$

PROBLEM 2

f is analytic in $\mathbb{C}$ and

$|f(z)| \leq |z|^n$ for some n

and large $|z|$

We shall show that if $m \geq n+1$

then $f^{(m)}(w) = 0$ for all

$w \in \mathbb{C}$

Now let R be very

large, then

$$f^{(m)}(w) = \frac{m!}{2\pi i} \int_{|z-w|=R} \frac{f(z)}{(z-w)^{m+1}} \, dz$$

Now

$$|f^{(m)}(w)| \leq \frac{m!}{2\pi} \int_{|z-w|=R} \frac{|f(z)|}{|z-w|^{m+1}} |dz|$$

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So

$$|f^{(m)}(w)| \le \frac{m!}{2\pi} \int_{|z-w|=R} \frac{|z|^n}{R^{m+1}} |dz|$$

Observe that

$|z| \ge |z-w| - |w| = R - |w|$ on the circle

So

$$|f^{(m)}(w)| \le \frac{m!}{2\pi} \int_{|z-w|=R} \frac{(R-|w|)^n}{R^{m+1}} |dz|$$

$$= \frac{m!}{2\pi} \frac{(R-|w|)^n}{R^m} \to 0$$

as $R \to \infty$ since

$m \ge n+1$

Problem 3:

f analytic in $\{z : |z| \le R\}$

and $|f(z)| \le M$ for all z

such that $|z| \le R$

Assume that $|z| \le \rho < R$

find an upper bound for

$|f^{(n)}(z)|$

Now.

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{|z|=R'} \frac{f(\xi)}{|\xi - z|^{n+1}} d\xi$$

For all $R' < R$

So
$$|f^{(n)}(z)| \le \frac{n!}{2\pi} \int_{|z|=R'} \frac{M}{|\xi - z|^{n+1}} |d\xi|$$

Now assume that $R' > \rho$,

then $|\xi - z| \ge |\xi| - |z|$

$= R' - |z| \ge R' - \rho$

since $|z| < \rho$

So

$$|f^{(n)}(z)| \leq \frac{n!}{2\pi} \int_{|z|=R'} \frac{M}{(R'-\rho)^{n+1}} |d\xi|$$

$$= \frac{n!}{2\pi} \frac{M}{(R'-\rho)^{n+1}} R' \, 2\pi$$

for all $R' < R$

$$= \frac{n! \, M R'}{(R'-\rho)^{n+1}}$$

let $R' \nearrow R$

we obtain that.

$$|f^{(n)}(z)| \leq \frac{n! \, M R}{(R-\rho)^{n+1}}$$