①

What have we covered so far

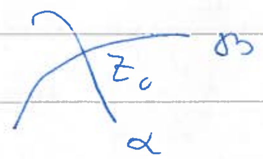
- Analytic functions

Ω open connected

f analytic if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$

exist for all $z_0 \in \Omega$

- Angel preserving
$f'(z_0) \neq 0$.



α, β arcs through z_0

$\Rightarrow$ angle between $f(\alpha)$ and $f(\beta)$ at $f(z_0)$ is the same as the angle between α and β at z_0

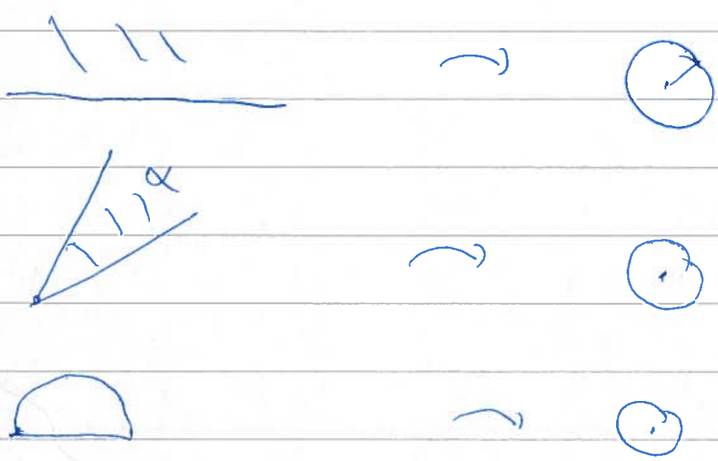
Chap 3

Linear fractional transformations

$$T(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0$$

T sends {lines and circles} to {lines and circles}

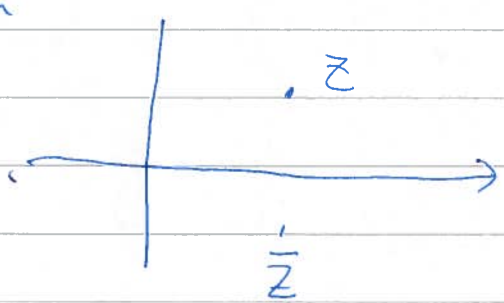
$$T^{-1}(w) = \frac{dw-b}{-cw+a}$$



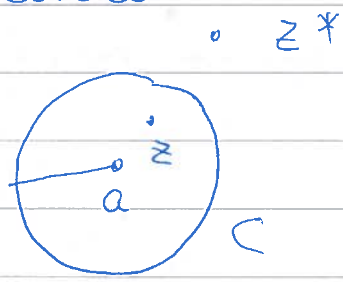
$$\frac{z-i}{z+i}$$

Reflection

over x-axis



over circles



$T : C \to$ x-axis

$$T(z^*) = \overline{T(z)}$$

Chap 4

Line integrals

$z(t)$, $a \leq t \leq b$

$$\int_\gamma f \, dz = \int_a^b f(z(t))\, z'(t)\, dt.$$

Cauchy's theorems:

4

1. Ω simply connected domain f analytic in Ω γ closed piecewise smooth curve.

1 if f' is continuous $\Rightarrow$

$$\int_\gamma f\,dz = 0$$

C-R equations + Greens theorem.

2. f analytic R a rectangle in Ω $\Rightarrow$

$$\int_{\partial R} f\,dz = 0$$

3. f analytic in Ω $\Rightarrow$

$\exists$ F such that $F' = f$

so F is analytic with continuous derivative.

4.

(5)

(4) If $\int_\gamma g\,dz = 0$ for

all piecewise smooth γ

and g is continuous

$\Downarrow$

γ simple and positive

$\cancel{n(\gamma, z)}$ $g(z) = \frac{1}{2\pi i} \int_\gamma \frac{g(\xi)}{\xi - z}\,d\xi$

AND

$$g^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{g(\xi)}{(\xi - z)^{n+1}}\,d\xi$$

also $g^{(n)}$ is continuous.

(5) Ω simply connected

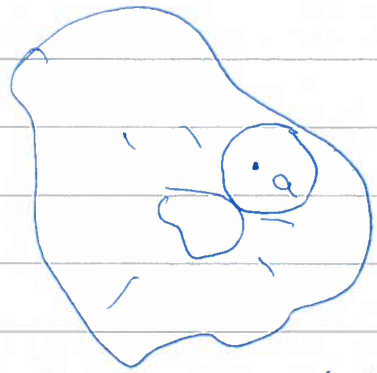
f is analytic in Ω

$\Longleftrightarrow$ $\int_\gamma f\,dz = 0$ for

all closed piecewise smooth curves γ.

- Liouvilles theorem :
f anal in $\mathbb{C}$ and $|f|$ is bounded
$\Rightarrow$ f is constant

Taylor series



f analytic in Ω

$a \in \Omega \Rightarrow$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n$$

in all $\Delta(a, r)$ s.t

$\Delta(a,r) \subset \Omega$.

- f analytic in Ω
  If there is an open $U \subset \Omega$
  such that $f|_U = 0$, then

  $f \equiv 0$ in Ω

- If f is analytic in Ω
  and f is nonconstant.
  Assume $f(z_0) = 0 \Rightarrow$
  $\exists\ r > 0$ such that
  $f(z) \neq 0$ if $z \in \Delta(z_0, r)$
  and $z \neq z_0$

Isolated singularities

Ω $\exists$ sequence of $\{a_j\}$
of descrete points and
f is analytic in $\Omega \setminus \{a_j\}$
We say that f have
isolated singularities.

$\underline{a}$ one of these

f analytic in

$$\Delta^*(a,r) = \Delta(a,r) \setminus \{a\}$$

(1) a is a removable singularity,

f is bounded in $\Delta^*(a,r)$
$\Rightarrow$ $\exists$ $\hat{f}$ anal in $\Delta(a,r)$
such that $f = \hat{f}$ in $\Delta^*(a,r)$

(2) a is a pole of order m
then $(z-a)^m f(z)$ is bounded
in $\Delta^*(a,r)$

(3) a is an Essential singularity
if is not as in (1) and (2)

4

- If f is an essential singularity then $f(\Delta^*(a,\delta))$ is dense in $\mathbb{C}$ for all δ.

LOCAL MAPPINGS
Page 130

- Theorem

f analytic in a simply connected Ω, γ closed curve in Ω $\Rightarrow$

$$\sum_j n(\gamma, z_j) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, dz$$

$\{z_j\}$ the zeros ~~for~~ of f

$n(\gamma, z_j)$ the winding number of γ around z_j

- A nonconstant analytic functions sends open sets to open sets.

- MAXIMUM PRINCIPLE

If f is analytic and nonconstant in Ω $\Rightarrow$ $|f(z)|$ does NOT have a max in Ω.

- Schwarz lemma

$f : \Delta(0,1) \to \Delta(0,1)$
$f(0) = 0$

Then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$

if equality at any point, then

$f(z) = \lambda z$ where $|\lambda| = 1$

Skip Setion 4
137 – 148.

Go to chap 5.
Section 1 page 175

- Weierstrass's theorem.

Ω, $\{f_n\}$ sequence of analytic function
If $f_n(z) \to f(z)$ uniformly on every compact in Ω, then f is analytic in Ω

- Laurent series

f analytic in $\{z : R_1 < |z-a| < R_2\}$

then $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$

$$\sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{-\infty}^{-1} b_n (z-a)^n$$

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$$\operatorname*{Res}_{z=a} f(z) = b_{-1}$$

TH $\quad f$ anal in $\Omega \setminus \{a_j\}$

Ω simply connected $\Rightarrow$

γ closed curve in Ω

$\Downarrow$

$$\frac{1}{2\pi i} \int_{\gamma} f(z)\,dz = \sum_j n(\gamma, a_j) \operatorname*{Res}_{z=a_j} f$$

ARGUMENT PRINCIPLE

f meromorphic

$\{z_s\}$ the zeros and

$\{a_j\}$ the poles of f

"

$$\frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)}\,dz = \sum_{s} n(\gamma, z_s) - \sum_{j} n(\gamma, a_j)$$

If no z_s or a_j is on γ

ROUCHES THEOREM

f, g analytic in Ω (s.c)

γ positive simple closed curve

$|f(z) - g(z)| < |f(z)|$.

Then $f(z)$ and $g(z)$ have

the same number of zeros

enclosed by γ