Project 1
Solutions

PROBLEM 1:

$$\frac{1}{2\pi i}\int_{|z|=4} \frac{\cos(\frac{\pi}{2}z)}{z(z-1)^2}\,dz = \frac{1}{2\pi i}\int_{|z|=4} f(z)\,dz$$

We see that the integrand have 2 singularities in the disc $\{z ; |z|<4\}$, one at $z_0=0$ and one at $z_1=1$, hence

$$\frac{1}{2\pi i}\int_{|z|=4} = \operatorname{Res}_{z=0} f(z) + \operatorname{Res}_{z=1} f(z)$$

Near $z=0$ we can write

$$f(z) = \frac{\frac{\cos(\frac{\pi}{2}z)}{(z-1)^2}}{z}$$ where $\frac{\cos(\frac{\pi}{2}z)}{(z-1)^2}$ is

analytic, further f have a simple pole at $z=0$ so

$$\operatorname{Res}_{z=0} f(z) = \frac{\cos(\frac{\pi}{2}0)}{(0-1)^2} = 1$$

near $z=1$ it might look like $f(z)$ have a double pole but

$\cos(\frac{\pi}{2}z)=0$ when $z=1$

If we power series $\cos(\frac{\pi}{2}z)$ near $z=1$, then

$$\cos(\tfrac{\pi}{2}z)=\cos\tfrac{\pi}{2} - \tfrac{\pi}{2}\sin(\tfrac{\pi}{2})(z-1)$$

$$+\sum_{n=2}^{\infty} a_n (z-1)^2$$

So near $z=1$

$$f(z) = \frac{\frac{1}{z}\left(-\frac{\pi}{2}(z-1)+\sum_{n=2}^{\infty} a_n (z-1)^2\right.}{(z-1)^2}$$

$$=\frac{1}{z}\left(-\frac{\pi}{2}\frac{1}{z-1}+\sum_{n=2}^{\infty} a_n (z-1)^{n-2}\right)$$

$$=-\frac{\pi}{2}\frac{\frac{1}{z}}{(z-1)} + \frac{1}{z}\sum_{n=2}^{\infty} a_n (z-1)^{n-2}$$

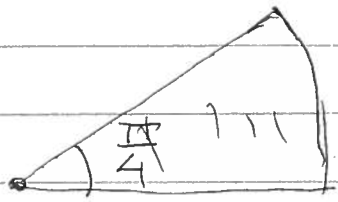
where $\frac{1}{z}\sum_{n=2}^{\infty} a_n (z-1)^{n-2}$

so $\underset{z=1}{\text{Res}}\, f(z) = -\frac{\pi}{2}\frac{1}{1} = -\pi/2.$

Hence $\frac{1}{2\pi i}\int \frac{\cos(\frac{\pi}{2}z)}{z(z-1)^2} = 1 - \frac{\pi}{2}$

PROBLEM 2

$D = \{z : 0 < |z| < 1$ and $0 < \arg z < \frac{\pi}{4}\}$



We want to find a conformal map from D to the upper half plane $\{z : \operatorname{Im} z > 0\}$

Some of you tried to go directly with a linear fractional transformation either to the upper half plane or the first quadrant. THIS WILL NOT WORK

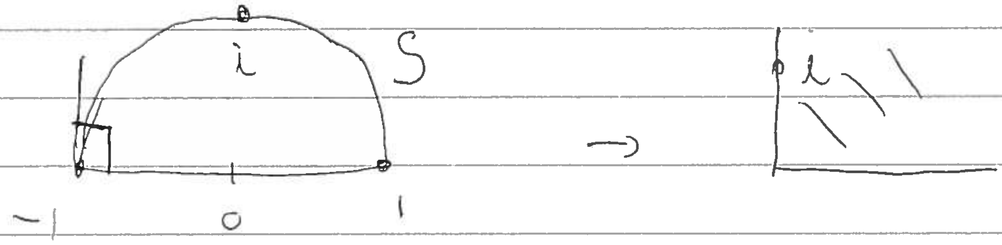
We start by opening D up to become $\{z : |z| < 1, \operatorname{Im} z > 0\}$

Use $\varphi_1(z) = z^4$

This is injective in D since $0 < \arg z < \frac{\pi}{4}$ when $z \in D$

Next we send

$\{|z| : |z|<1, \operatorname{Im} z>0\}$ to the first quadrant



Here we can use a linear fractional transformation

$$T(z) = \frac{az+b}{cz+d}$$

$-1 \to 0$, $0 \to 1$ and $1 \to \infty$

$$T(z) = \frac{z+1}{1-z}$$

Now the upper half circle will become a ray and the line $[-1,1]$ will go to the positive real axis

$T(S)$ will be $90°$ degrees on $T([-1,1])$ since S and $[-1,1]$ makes an angle of $90°$ at the point -1.

From this we get that

$T\{z;\ |z|<1,\ \operatorname{Im} z>0\}$ is either the first or the fourth quadrant

Test by finding

$$T(i) = \frac{i+1}{-1} = \frac{(i+1)^2}{2}$$

$$= \frac{-1+2i+1}{2} = i$$

So the image is the first quadrant.

Finally we use $\varphi_2(z) = z^2$

Then $\psi(z) = \varphi_2 \circ T \circ \varphi_1(z)$

$$= \left(\frac{z^4+1}{1-z^4}\right)^2$$

will send D to $\{z:\ \operatorname{Im} z>0\}$

PROBLEM 3

f analytic in a domain Ω and there is a point $a \in \Omega$ such that $f^{(n)}(a) = 0$ for all $n = 0, 1, 2, \ldots$

Let $\Delta(a, r) \subset \Omega$, $r > 0$, then Taylor's theorem for analytic functions implies that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n = 0$$

when $z \in \Delta(a, r)$

Note that $\Delta(a, r)$ is open so we can use the theorem:

THEOREM

Assume that Ω is a domain (open and connected) if f is analytic in Ω and there is an open $U \subset \Omega$ such that $f|U \equiv 0$, then $f(z) = 0$ for ALL $z \in \Omega$.

PROBLEM 4:

a) Let $\alpha \in \mathbb{C}$ and $|\alpha| < 1$, then

$\phi_\alpha(z) = \dfrac{z-\alpha}{1-\bar{\alpha} z}$ sends α to 0

the unit circle to the unit circle and the unit disc to the unit disc.
There are several ways to do this, I will choose one of them.

① $\phi_\alpha(\alpha) = \dfrac{\alpha-\alpha}{1-\bar{\alpha}\alpha} = 0$

We can see that if $|z| \leq 1$, then $|\bar{\alpha} z| \leq |\bar{\alpha}| < 1$ so ϕ_α do not have any singularities on $\Delta(0,1)$.

② If z is on the unit circle then $z = e^{i\theta}$ for some θ so

$$|\phi_\alpha(z)| = |\phi_\alpha(e^{i\theta})| = \left| \frac{e^{i\theta}-\alpha}{1-\bar{\alpha} e^{i\theta}} \right|$$

$$= \frac{1}{|e^{i\theta}|} \left| \frac{e^{i\theta}-\alpha}{e^{-i\theta}-\bar{\alpha}} \right| = \left| \frac{e^{i\theta}-\alpha}{\overline{(e^{i\theta}-\alpha)}} \right| = 1.$$

(3)

The unit circle S is the boundary of 2 domains

(a) $\{\xi ; |\xi| < 1\}$

(b) $\{\xi ; |\xi| > 1\}$

So $\phi_\alpha(\Delta(0,1))$ is one of these since

$\alpha \in \Delta(0,1)$ and $\phi_\alpha(\alpha) = 0 \in \Delta(0,1)$

it follows that

$\phi_\alpha(\Delta(0,1)) = \Delta(0,1)$

ii

Assume that f is analytic in $\Delta(0,1)$ and $|f(z)| < 1$ for all $z \in \Delta(0,1)$.

If there are two points α and β such that $\alpha \neq \beta$ and $f(\alpha) = \alpha$ and $f(\beta) = \beta$, then $f(z) = z$ for all $z \in \Delta(0,1)$

STEP 1 Assume that $\alpha = 0$

Then Schwarz lemma implies that $|f(z)| \leq |z|$ and if there is a $z_0 \neq 0$ such that $|f(z_0)| = |z_0|$, then there is a λ where $|\lambda| = 1$ and

$$f(z) = \lambda z \quad \text{for all } z$$

Let $z_0 = \beta \neq 0 \Rightarrow |f(\beta)| = |\beta|$ so $f(z) = \lambda z$ for all z, but

$f(\beta) = \beta$ so $\lambda = 1$,

STEP2:

Now assume that $\alpha \neq 0$ and $\beta \neq 0$.

Let $g(z) = \Phi_\alpha \circ f \circ \Phi_\alpha^{-1}$, then

$$g(0) = \Phi_\alpha \circ f \circ \Phi_\alpha^{-1}(0)$$

$$= \Phi_\alpha (f(\alpha)) =$$

$$\Phi_\alpha(\alpha) = 0 \quad \text{so}$$

$$\underline{g(0) = 0}$$

We need to show that there is a β' such that

$$g(\beta') = \beta' \quad , \beta' \neq 0$$

Let $\beta' = \Phi_\alpha(\beta)$ since $\beta \neq \alpha$ and Φ_α is injective, then $\beta' \neq 0$

and

$$g(\beta') = \Phi_\alpha (f (\Phi_\alpha^{-1}(\beta')))$$

$$= \Phi_\alpha(f(\beta)) = \Phi_\alpha(\beta) = \beta'$$

From STEP 1 it follows that $g(z)=z$ so

$$\phi_\alpha(f(\phi_\alpha^{-1}(z)))=z$$

$$\Downarrow$$

$$f(\phi_\alpha^{-1}(z)) = \phi_\alpha^{-1}(z)$$

$$\Downarrow$$

$$f(z) = \phi_\alpha^{-1}\phi_\alpha(z) = z.$$

PROBLEM 5

There are two ways to prove the statement

METHOD 1:

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

compare P to $f(z)=z^n$

Let R be large and look at

$$|P(z) - f(z)| \text{ on } \{z : |z| = R\}$$

Now

$$|P(z) - g(z)| = |a_{n-1} z^{n-1} + \cdots + a_1 z + a_0|$$

$$= |z|^{n-1} \left| a_{n-1} + \frac{a_{n-2}}{z} + \cdots \frac{a_1}{z^{n-2}} + \frac{a_0}{z^{n-1}} \right|$$

$$\leq |z|^{n-1} \left(|a_{n-1}| + |a_{n-2}| + \cdot + |a_0| \right)$$

if $|z| = R$ and $R \geq 1$

So $|P(z) - g(z)| \leq R^{n-1} \sum_{i=0}^{n-1} |a_i|$

när $|z| = R$

Choose R so large

that $\sum_{i=0}^{n-1} |a_i| < R$, then

$|P(z) - g(z)| < R^n = |g(z)|$ when $|z| = R$

g have a root of multiplicity n at 0 so $P(z)$ have n roots in $\Delta(0, R)$ from Rouche's theorem, since R can be arbitrary large it follows that P have n roots in $\mathbb{C}$

METHOD 2:

We will use induction on n.

First we need to show that if $n \geq 1$, then P must have at least 1 root.

Proof:

Assume not, then

$\frac{1}{P(z)}$ is analytic in $\mathbb{C}$

As we have seen

$$\lim_{|z| \to \infty} \frac{1}{P(z)} = 0$$

Hence there exist an R such

that $\left|\frac{1}{P(z)}\right| < 1$ if $|z| \geq R$

But $\frac{1}{P(z)}$ is continuous so

$\left|\frac{1}{P(z)}\right| \leq C < \infty$ on $\overline{\Delta(0,R)}$ for some C

Liouvilles theorem implies that

$\frac{1}{P}$ hence P is constant

But if $n \geq 1$ then P is not constant.

If $n = 1$ then $P(z) = a_1 z + a_0$
where $a_1 \neq 0$ so

$z = -\frac{a_0}{a_1}$ is a root for P

Assume that we have established that all degree k polynomials have k roots, counting multiplicity, for $k \leq n-1$

We know that our degree n polynom have at least one root z_1

If we can show that

$Q(z) = \frac{P(z)}{z - z_1}$ is a degree $n-1$ polynom, then $P(z) = (z - z_1)\, Q(z)$

and Q have $n-1$ roots.

P have n roots.

We can rewrite $P(z)$

$$P(z) = (z - z_1 + z_1)^n + a_{n-1}(z - z_1 + z_1)$$

$$+ \dots + a_1(z - z_1 + z_1) + a_0$$

We use the binomial formula and obtain:

$$P(z) = (z - z_1)^n + b_{n-1}(z - z_1)^{n-1} + \dots b_1(z - z_1) + b_0$$

but $P(z_1) = b_0 = 0$

so

$$P(z) = (z - z_1)\left[(z - z_1)^{n-1} + \dots + b_1\right]$$

$$= (z - z_1)\, Q(z)$$