

## Exercises Week 12

1. Let  $a, \lambda \in \mathbb{C}$  with  $|a| < 1$  and  $|\lambda| = 1$ . Show that  $\phi: \mathbb{D} \rightarrow \mathbb{C}$ , given by

$$z \mapsto \lambda \frac{z - a}{1 - \bar{a}z},$$

maps  $\mathbb{D}$  biholomorphically to itself and maps  $\{z \in \mathbb{C}: |z| = 1\}$  bijectively to itself.

2. Show that any biholomorphism  $\phi: \mathbb{D} \rightarrow \mathbb{D}$  is a map of the kind studied in the previous exercise. In particular, every biholomorphism  $\mathbb{D} \rightarrow \mathbb{D}$  is given by a Möbius transformation.
3. According to the Riemann mapping theorem, there exist biholomorphisms  $\{z \in \mathbb{C}: \operatorname{Re}(z) > 0, \operatorname{Im}(z) > 0\} \rightarrow \mathbb{D}$ . Give one such biholomorphism explicitly. *Hint:* First map to a half plane, then use Möbius transformations.
4. Let  $R$  be a positive real number. According to the Riemann mapping theorem, there exist biholomorphisms  $\{z = re^{i\theta} \in \mathbb{C}: r \in (0, R) \text{ and } \theta \in (0, \pi/10)\} \rightarrow \mathbb{D}$ . Give one such biholomorphism explicitly.