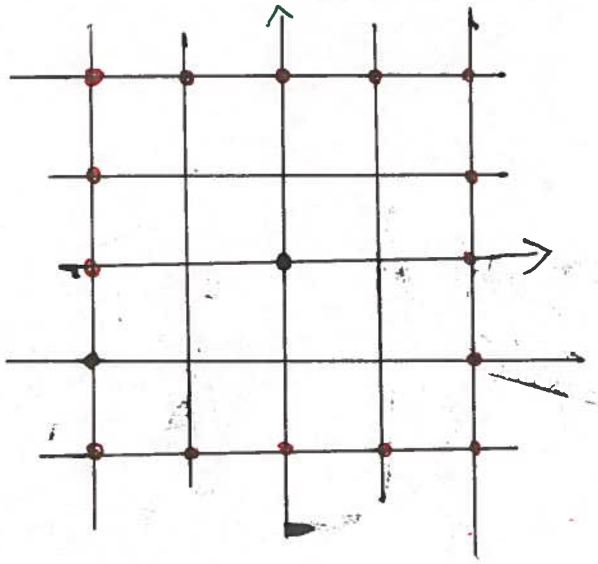
COMPLEX ANALYSIS

Peter Lindqvist

29. V. 2019

TMA 4175.

(1)



We show that the series is absolutely convergent. There are

$8k$

points (m,n) in the k'th frame

$|m| = k$ or $|n| = k$, and $|m| \le k$, $|n| \le k$

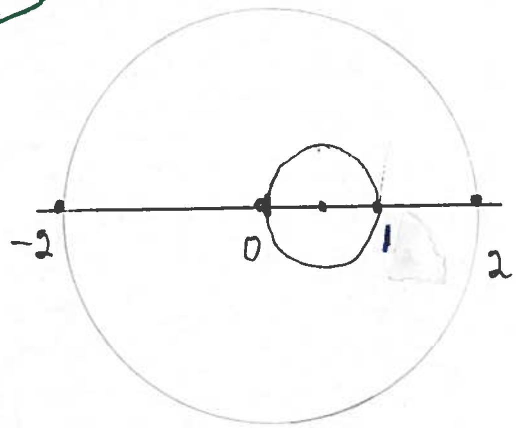
$$\sum_{m,n}{}' \left|\frac{1}{m+in}\right|^3 = \sum_{m,n}{}' \frac{1}{(m^2+n^2)^{3/2}}$$

$$\le \sum_{k=1}^{\infty} 8k \cdot \frac{1}{k^3} = 8\sum_{k=1}^{\infty}\frac{1}{k^2} < \infty$$

An absolutely conv. series is convergent. (In fact, $\sum_{m,n}{}'(m+in)^{-3} = 0$.)

(2) The periodic function is bounded in the square $|x| \le 2$, $|y| \le 2$, say. Then it is bounded in the whole plane $\mathbb{C}$. By Liouville's Theorem it reduces to a constant.

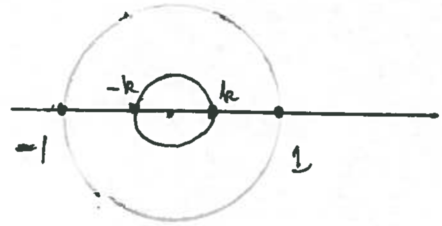
3



$|z| = 2$

$|z - \frac{1}{2}| = \frac{1}{2}$

concentric circles.



$|w| = k$

$|w| = 1$

Use a Möbius transformation that maps $z \longrightarrow w = w(z)$ and

$$\begin{cases} -2 \longrightarrow -1 \\ 2 \longrightarrow 1 \\ 0 \longrightarrow -k \\ 1 \longrightarrow +k \end{cases}$$

$$\frac{w-1}{w+1} = K \, \frac{z-2}{z+2}$$

$$\frac{-k-1}{-k+1} = -\frac{2K}{2} = -K \qquad \frac{1+k}{1-k} = K > 0$$

$$\frac{k-1}{k+1} = K \frac{1-2}{1+2} = -\frac{K}{3} \qquad K^2 = 3, \quad K = +\sqrt{3}$$

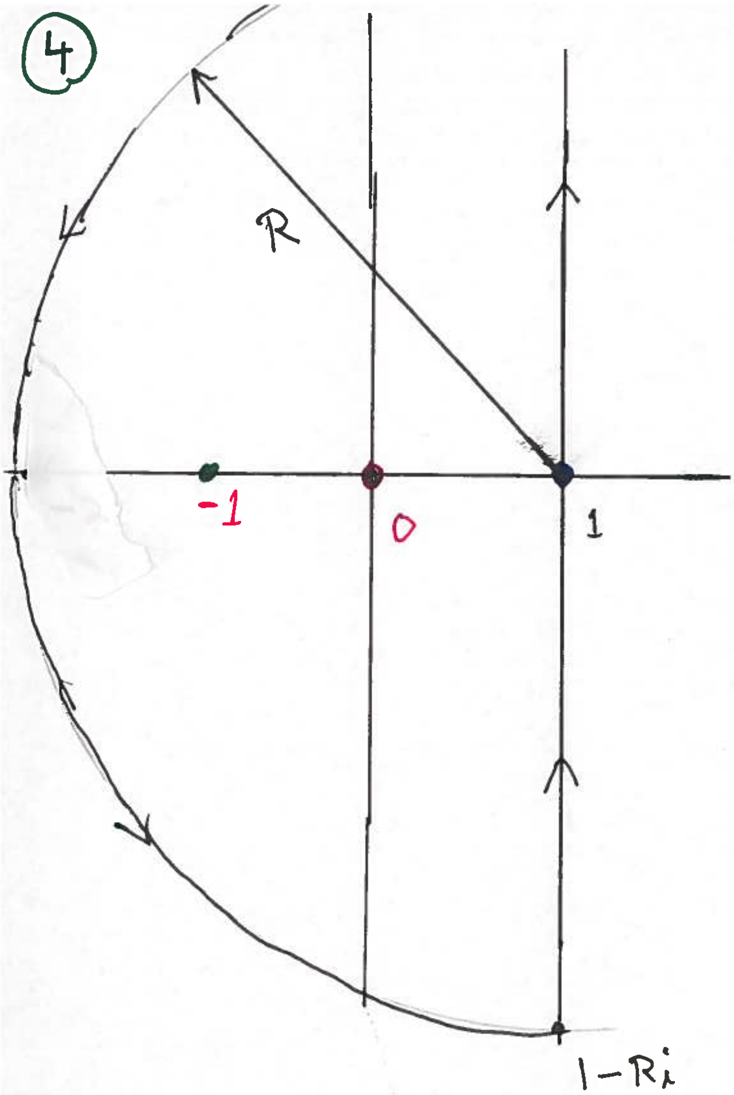
$$1 + k = \sqrt{3}\,(1-k) \implies k = 2 - \sqrt{3}$$

In fact,

$$\left(w = \frac{(1+\sqrt{3})z + 2(1-\sqrt{3})}{(1-\sqrt{3})z + 2(1+\sqrt{3})}. \right)$$

Answer: $R = 2 - \sqrt{3}$

④



The contour is the half-circle

$$z = 1 + Re^{i\theta}, \quad \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$$

and its diameter $[1-Ri, 1+Ri]$. The residues at the simple poles are

$$\operatorname{Res}_{z=0}\left\{\frac{2^z}{z(z+1)}\right\} = \frac{2^0}{0+1} = 1$$

$$\operatorname{Res}_{z=-1}\left\{\frac{2^z}{z(z+1)}\right\} = \frac{2^{-1}}{-1} = -\frac{1}{2}$$

Thus

Residue Theorem.

$$1 - \frac{1}{2} = \frac{1}{2\pi i}\oint \frac{2^z\,dz}{z(z+1)} = \frac{1}{2\pi i}\int_{1-Ri}^{1+Ri} \frac{2^z\,dz}{z(z+1)}$$

$$+ \underbrace{\frac{1}{2\pi i}\int_{\pi/2}^{3\pi/2} \frac{e^{(1+Re^{i\theta})\ln 2}\, iRe^{i\theta}}{(1+Re^{i\theta})(2+Re^{i\theta})}\,d\theta}_{\equiv I_R}$$

The integral along the arc approaches zero.

$$|I_R| \le \frac{1}{2\pi}\cdot\pi\, \frac{e^{(1+R\cos\theta)\ln 2}\,R}{(R-1)(R-2)} \le \frac{1}{2}\,\frac{e^{\ln 2}R}{(R-1)(R-2)} \xrightarrow{R\to\infty} 0.$$

Answer: $1 - 1/2 = 1/2$.

5 $$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt = n^s \int_0^\infty e^{-nx} x^{s-1} dx$$

$$\boxed{s > 1}$$

$t = nx$
$dt = n\,dx$

$$\Gamma(s) \sum_{n=1}^\infty n^{-s} = \sum_{n=1}^\infty \int_0^\infty e^{-nx} x^{s-1} dx$$

$$\overset{*)}{=} \int_0^\infty \underbrace{\left(\sum_{n=1}^\infty e^{-nx}\right)}_{\frac{1}{e^x - 1}} x^{s-1} dx = \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx$$

(Geometric Series)

---

*) The series is absolutely convergent for $x > 0$.

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx$$

Riemann's Zeta.

⑥ $\cos(z) = 0 \iff z = \frac{\pi}{2} + n\pi\ ,\quad n = 0, \pm 1, \pm 2, \ldots$

$$|\cos(z)| = \frac{1}{2}\left|e^{iz} + e^{-iz}\right| \le \frac{1}{2}\left(|e^{iz}| + |e^{-iz}|\right)$$

$$\le e^{|z|} \quad (\text{of Growth Order } 1)$$

Thus $g(z) = a + bz$ (Hadamard)

$$\cos(z) = e^{a+bz} \prod_{n=-\infty}^{\infty} \left(1 - \frac{2z}{\pi(1+2n)}\right) e^{\frac{2z}{\pi(1+2n)}}$$

- The convergence is evident since

$$\left(1 - \frac{2z}{\pi(1+2n)}\right) e^{2z/\pi(1+2n)} = 1 - \frac{4z^2}{\pi^2(1+2n)^2} + O\left(\frac{1}{n^3}\right)$$

and $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(1+2n)^2}$ converges.

- $z = 0$ yields $a = 0$
- Combining the terms with n and $-(1+n)$ we get the convergent product

$$\cos(z) = e^{bz} \prod_{n=0}^{\infty} \left(1 - \frac{4z^2}{\pi^2(1+2n)^2}\right)$$

Now $\cos(z) = \cos(-z)$ yields $\underline{b = 0}$.

$$\boxed{\cos(z) = \prod_{n=0}^{\infty} \left(1 - \frac{4z^2}{(1+2n)^2\pi^2}\right)}$$