EX.: Let $a_n > 0$, $n = 1, 2, 3, \ldots$ Show that

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \le \liminf_{n\to\infty} \sqrt[n]{a_n} \le$$

$$\limsup_{n\to\infty} \sqrt[n]{a_n} \le \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

In particular, if $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists so does $\lim\limits_{n\to\infty} \sqrt[n]{a_n}$ and they are equal.

Recall that $\zeta = \limsup\limits_{n\to\infty} b_n$ if, given $\varepsilon > 0$ there is an index N_ε such that

1°) $b_n < \zeta + \varepsilon$, when $n > N_\varepsilon$

2°) $b_n > \zeta - \varepsilon$ for some b_n, $n > N_\varepsilon$.

Hint.

$$a_n = \left(a_1 \frac{a_2}{a_1} \frac{a_3}{a_2} \cdots \frac{a_{N_\varepsilon+1}}{a_{N_\varepsilon}}\right) \cdot \frac{a_{N_\varepsilon+2}}{a_{N_\varepsilon+1}} \cdots \frac{a_n}{a_{n-1}}$$